

ELECTRIC CIRCUIT ANALYSIS

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BY

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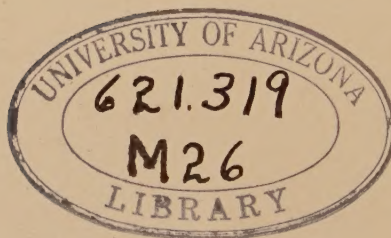
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PREFACE

The material presented in this book is intended to acquaint the student of electrical engineering with the principles and laws of electric circuits.

The student is called upon, in this work, to make use of certain branches of mathematics with which he is not quite thoroughly familiar. Accordingly the first three chapters are devoted to the study of complex numbers, vectors, and the theory of sine functions.

There are three mathematical ways of representing sine Emfs. and currents: (a) instantaneous values, (b) vectors, (c) complex numbers. These three modes of representation are, however, intrinsically related. The method of transforming an expression from one of these forms into the other is given in Chapter III. In the subsequent treatment it is only sufficient to express the physical facts in any one of these mathematical forms, derive the desired result, and, then, transform this result into the other two forms. This method possesses not only the merit of economy in mental effort but also the advantage of leading the student to a unified point of view.

Although physical laws are independent of the choice of physical units, nevertheless the application of these laws to numerical problems requires the use of a consistent set of units. Chapter IV is therefore, devoted to the study of electric units and their dimensions.

The reaction of a circuit to an impulse is determined by certain properties (characteristics) of that circuit. It is essential to understand the physical nature of these characteristics before undertaking the study of electric circuits. The fifth chapter is therefore devoted to the study of the characteristics of an electric circuit.

The phenomena occurring in an electric circuit may be either transient or steady. Steady phenomena admit of simple mathematical analysis. Transient phenomena, however, require comparatively more involved mathematics. Experience shows that a student, having once mastered the analysis of steady phenomena, is better prepared to understand transients. Hence the first part of the text (Chapters VI to XVI) is devoted to the study of circuits in the steady state. The consideration of circuits in the transient state is reserved to a later part of the text (Chapters XVII and XVIII).

The treatment (Chapter IX) of circuits with variable characteristics through the use of complex quantities possesses generality, simplicity, and brevity. It is unfortunate that this mode of treatment of such circuits is not more generally adopted.

No apology is made for including Chapter XIII on Fourier's Series. The electrical engineer is called upon to analyze a non-sine wave into its component harmonics — a task which is tedious at best. Hence any efforts expended in lightening his task should be encouraged. The author realizes, however, that, in general, the technical colleges do not desire as exhaustive a treatment of harmonic analysis as is given in Chapter XIII. On the other hand they expect to cover the theory of circuits subjected to non-sine Emfs. In order to satisfy the needs of such colleges, the treatment of Chapter XIV is made independent of Chapter XIII.

The mathematical analysis of any physical problem is subject to a clear understanding of the physical phenomena involved. Hence, throughout the text, a description of the physical phenomena always precedes any mathematical analysis thereof.

The various chapters are made as independent of one another as possible, and the material covered in each chapter is so graded that any chapter or part thereof may be omitted without impairing the continuity of the text.

Experience has shown that quantitative relationships cannot be mastered without a liberal use of illustrative problems. Many problems have therefore been incorporated in the text. They are numbered according to the articles which they are intended to elucidate; for example, the problems 2a, 2b, etc. all illustrate Art. 2. The answers to these problems in the form of a separate pamphlet can be secured from the publisher.

The text is illustrated by numerous figures which are so arranged that a student can readily correlate a diagram of connections with the corresponding sine waves and vector diagram.

Literature and text references are, except in a few instances, omitted. The reader who is interested in any one topic may consult Science Abstracts for articles and the catalogues of any engineering library or publishing company for texts.

Previous to its publication, this book was used in mimeographed form both at Cornell University and at Northeastern University. I am indebted to my former colleague Mr. D. D. MacCarthy and to Prof. Wm. Lincoln Smith of Northeastern University for some helpful suggestions. Thanks are also due to some professors in the department of mathematics at Cornell University and particularly to Prof. J. I.

Hutchinson whose courtesy and helpfulness I keenly appreciate. I am also indebted to Mr. A. C. Stevens of the Educational Dept. of the General Electric Co. for placing at my disposal the facilities of that department; and to the publishers and compositors for their courtesy, accuracy, and dispatch in the difficult composition of this text. Finally I feel particularly grateful to Prof. V. Karapetoff who read the mimeographed text and offered many valuable comments, criticisms, and suggestions.

MICHEL G. MALTI

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CONTENTS

CHAPTER	PAGE
I. COMPLEX NUMBERS.....	1
II. VECTORS.....	14
III. SINE FUNCTIONS.....	21
IV. PHYSICAL QUANTITIES, UNITS, AND DIMENSIONS.....	37
V. THE CHARACTERISTICS OF AN ELECTRIC CIRCUIT.....	46
VI. SIMPLE ALTERNATING ELECTRIC CIRCUITS.....	58
VII. SERIES CIRCUITS WITH CONSTANT CONCENTRATED CHARACTERISTICS.....	72
VIII. PARALLEL CIRCUITS WITH CONSTANT CONCENTRATED CHARACTERISTICS.....	91
IX. SERIES OR PARALLEL CIRCUITS WITH VARIABLE CONCENTRATED CHARACTERISTICS.....	106
X. SERIES-PARALLEL COMBINATIONS OF CIRCUIT CHARACTERISTICS AND OF ALTERNATING EMF. SOURCES.....	121
XI. COUPLED CIRCUITS.....	135
XII. ENERGY AND POWER IN ALTERNATING ELECTRIC CIRCUITS.....	147
XIII. FOURIER'S SERIES AND HARMONIC ANALYSIS.....	168
XIV. CIRCUITS WITH NON-SINE PERIODIC EMFS. AND CURRENTS.....	204
XV. POLYPHASE CIRCUITS, PART A — BALANCED CIRCUITS.....	225
XVI. POLYPHASE CIRCUITS, PART B — UNBALANCED CIRCUITS.....	258
XVII. ELECTRIC TRANSIENTS, PART A — SINGLE-ENERGY TRANSIENTS ...	281
XVIII. ELECTRIC TRANSIENTS, PART B — DOUBLE-ENERGY TRANSIENTS ..	301
XIX. CIRCUITS WITH UNIFORMLY DISTRIBUTED CHARACTERISTICS.....	317
APPENDIX	
I. THEOREMS USED IN THE TEXT AND THEIR PROOFS.....	344
II. MATHEMATICAL AND PHYSICAL TABLES.....	349
INDEX.....	379

ELECTRIC CIRCUIT ANALYSIS

CHAPTER I

COMPLEX NUMBERS

1. Imaginary Numbers. — The solution of the equation $x^2 + a^2 = 0$ is:

$$\dot{x} = \pm \sqrt{-a^2} = \pm a \sqrt{-1} \quad (1)$$

This number $\sqrt{-1}$ is called the *imaginary unit* and is denoted by the letter j . Substituting $j = \sqrt{-1}$ in Eq. (1) we have:

$$\dot{x} = \pm ja \quad (2)$$

The number $\dot{x}$ is called an *imaginary* or *j-number* because it contains the imaginary unit, j , as a factor. The term imaginary is not at all descriptive of this type of numbers. Indeed they are just as real as are the ordinary real numbers. Imaginary numbers might have been called *j-numbers* with less confusion and more propriety. We shall, therefore, refer to them interchangeably as *j-numbers* or *imaginary numbers*. A dot placed under a letter ($\dot{x}$) signifies that such a letter represents an imaginary number.

2. The Periodicity of j^n . — By definition, $j = \sqrt{-1}$. Hence

$j^2 = -1$; $j^3 = -j$; $j^4 = 1$, etc. we thus conclude that:

$$\left. \begin{aligned} j^{4n} &= 1 & (a) \\ j^{(4n+a)} &= j^{4n}j^a = j^a & (b) \end{aligned} \right\} \quad (3)$$

where n is any real integer.

3. Complex Numbers. — The solution of the equation $Y^2 - 2Y + 5 = 0$ is:

$$\mathbf{Y} = (2 \pm \sqrt{4 - 20})/2 = 1 \pm j2 \quad (4)$$

The number $\mathbf{Y}$ is neither a *j-number* nor a real number but a combination of both. Such a number is called a *complex number*. In general a complex number $\mathbf{A} = a + ja'$ is a number consisting of a real term, a , and an imaginary term, ja' , a bold-face capital letter $\mathbf{A}$ is used to represent complex numbers. This notation is selected merely for convenience

and brevity in writing complex numbers as it represents both the real and j -terms by just one letter.*

The real-term and the j -term of the complex number $a + ja'$ are two separate and distinct entities. Hence the sign of addition or subtraction placed between them does not convey the same meaning that it does when used with two real numbers. To illustrate this fact let it be required to add six houses and five men. The answer is simply 6 houses + 5 men — not eleven. Similarly the real- and j -terms of a complex number are each an entity. The two terms can neither be added together nor subtracted from each other. The sign between the terms has a graphic significance which will be explained in the next article.

4. Graphic Representation of Complex Numbers. — The real- and j -terms of a complex number may be either positive, negative, or zero. Hence a complex number may assume any one of the following forms:

$$\left. \begin{array}{ll} (a) \mathbf{A}_1 = a + ja' & (b) \mathbf{A}_2 = a - ja' \\ (c) \mathbf{A}_3 = -a - ja' & (d) \mathbf{A}_4 = -a + ja' \\ (e) \mathbf{A}_5 = 0 \pm ja' & (f) \mathbf{A}_6 = \pm a + j0 \end{array} \right\} \quad (5)$$

Any of the complex numbers given in Eq. (5) may be represented graphically on a system of rectangular coördinates (Fig. 1) where the real part is plotted along the horizontal axis called the *axis of reals* or the *R-axis*, and the imaginary part is plotted along a vertical axis called the *axis of imaginaries* or the *j-axis*. Thus let the real part of a complex number be plotted along the *R-axis* and to the right of the point O (Fig. 1) if positive — or to the left of O if negative. Similarly

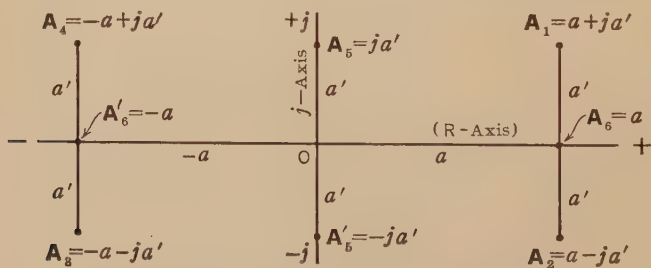


FIG. 1. Graphic representation of complex numbers.

let the j -part be plotted along the j -axis — above the origin O if positive and below it if negative. Then the complex numbers given in Eq. (5) may be represented by the points $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_6$ respectively.

* While in print it is easy to distinguish between bold-face and ordinary type, it becomes difficult to do so in script. It is convenient to use other notations for script purposes. The following notations are used $\hat{A}$; $\check{A}$; $\ddot{A}$.

Fig. 1 gives a graphic interpretation of the meaning of the $+$ or $-$ sign preceding the real- and j -terms of a complex number. These signs are not signs of addition or subtraction but rather symbols which indicate whether the complex number lies in the first, second, third or fourth quadrant of the rectangular system of coördinates.

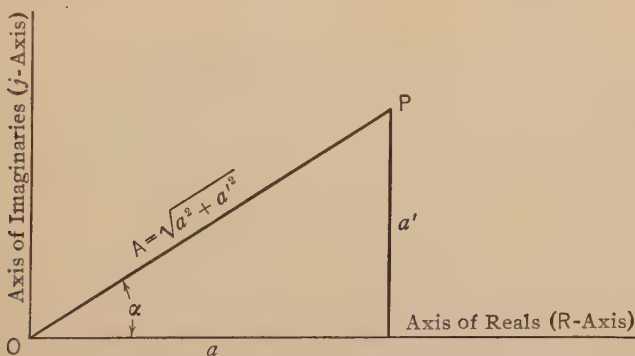
5. Forms of Expressing Complex Numbers. — A complex number may be expressed in any one of the following forms all of which are identical:

- a. Vectorial Form.
- b. Orthogonal Form.
- c. Trigonometric or Circular Form.
- d. Exponential or Polar Form.

(a) *Vectorial Form.* — A complex number is said to be expressed in the vectorial form when it is represented by just one letter. Many forms are in use of which the following may be cited $\dot{A}$, $\check{A}$, $\hat{A}$, $\mathbf{A}$. We shall use exclusively the first and last forms viz. a capital letter with a dot ($\dot{A}$) and a bold-face capital letter ($\mathbf{A}$).

(b) *Orthogonal Form.* — A complex number is said to be expressed in the orthogonal form when the actual magnitudes of its real- and j -terms are given. Thus $\mathbf{A} = a + ja'$.

(c) *Trigonometric or Circular Form.* — Let the point P (Fig. 2) represent the complex number $a + ja'$ plotted in rectangular coördinates.



The line OP has a length

$$A = \sqrt{a^2 + a'^2} \quad (6)$$

and makes with the R -axis an angle α such that:

$$\tan \alpha = (a'/a). \quad (7)$$

The length $OP = A$ is called the *modulus** of the complex number and the angle α , measured counterclockwise from the axis of reals, is termed its *argument*.

The following trigonometric relations follow from Fig. 2.

$$\left. \begin{aligned} a &= A \cos \alpha & (a) \\ a' &= A \sin \alpha & (b) \end{aligned} \right\} \quad (8)$$

Hence the complex number:

$$\mathbf{A} = a + ja' = A (\cos \alpha + j \sin \alpha). \quad (9)$$

When a complex number is expressed, as in Eq. (9), in terms of its modulus and the cosine and sine of its argument, that complex number is said to be expressed in the trigonometric form.

Variations of Eq. (9) are:

$$\mathbf{A} = A (\cos \alpha + j \sin \alpha) = A \operatorname{cis} \alpha = A \angle \alpha \quad (10)$$

The term $\operatorname{cis} \alpha$ is merely an abbreviation of $(\cos \alpha + j \sin \alpha)$. The term $A \angle \alpha$ is a shorthand way of stating that the complex number has a modulus A and an argument α .

(d) *Exponential or Polar Form*:—When a complex number $\mathbf{A}$ is expressed in the form $A\epsilon^{j\alpha}$, it is said to be expressed in the exponential or polar form. We shall now prove the identity of the expressions $A\epsilon^{j\alpha}$ and $A(\cos \alpha + j \sin \alpha)$. Let:—

$$\mathbf{Y} = \cos \alpha + j \sin \alpha \quad (11)$$

By differentiation, Eq. (11) becomes:

$$(d\mathbf{Y}/d\alpha) = -1 \sin \alpha + j \cos \alpha \quad (12)$$

Substituting (j^2) for (-1) in Eq. (12) we have:

$$(d\mathbf{Y}/d\alpha) = j^2 \sin \alpha + j \cos \alpha = j (\cos \alpha + j \sin \alpha) = j\mathbf{Y} \quad (13)$$

whence

$$(d\mathbf{Y}/\mathbf{Y}) = j d\alpha \quad (14)$$

Integrating Eq. (14) we obtain:

$$\ln \mathbf{Y} = j\alpha + K \quad (15)$$

By definition of a logarithm, Eq. (15) may be put in the following form:

$$\mathbf{Y} = \epsilon^{j\alpha+K} \quad (16)$$

Substituting the value of $\mathbf{Y}$ from Eq. (11) in Eq. (16) we have:

$$\cos \alpha + j \sin \alpha = \epsilon^K \epsilon^{j\alpha} \quad (17)$$

* The modulus of a complex number is represented by a capital letter which is *not bold-faced*.

In order that Eq. (17) may be an identity it must hold true for all values of α and, therefore, for the particular value $\alpha = 0$. Substituting this value in Eq. (17) and simplifying we obtain:

$$1 = e^{j0} \quad (18)$$

Substituting Eq. (18) in Eq. (17) and multiplying both sides by A we obtain:

$$A(\cos \alpha + j \sin \alpha) = A e^{j\alpha} \quad (19)$$

The identity of all the foregoing forms of expressing a complex number may be summarized by the following equation which will be found useful for reference:

$$A \equiv a + ja' \equiv A(\cos \alpha + j \sin \alpha) \equiv A \text{ cis } \alpha \equiv A \angle \alpha \equiv A e^{j\alpha} \quad (20)^*$$

6. Equality of Complex Numbers. — Two complex numbers are equal when their real parts are equal and their imaginary parts are equal. Thus the equation:

$$a + ja' = b + jb' \quad (21)$$

means:

$$\left. \begin{array}{l} a = b \quad (a) \\ a' = b' \quad (b) \end{array} \right\} \quad (22)$$

Changing Eq. (21) to the circular form we have:

$$A(\cos \alpha + j \sin \alpha) = B(\cos \beta + j \sin \beta) \quad (23)$$

where (see Eqs. 6 and 7):

$$\left. \begin{array}{l} A = \sqrt{a^2 + a'^2}; \quad B = \sqrt{b^2 + b'^2} \quad (a) \\ \tan \alpha = (a'/a); \quad \tan \beta = (b'/b) \quad (b) \end{array} \right\} \quad (24)$$

But by Eq. (22) $a = b$ and $a' = b'$. Hence:

$$A = B, \quad \alpha = \beta \quad (25)$$

Eq. (25) states that when two complex numbers are equal then their moduli are equal and their arguments are equal.

7. Addition and Subtraction of Complex Numbers. — The addition of complex numbers is performed by separately adding the real- and j -terms. Thus:

$$(a + ja') + (b + jb') = (a + b) + j(a' + b') \quad (26)$$

* It is very convenient, indeed sometimes indispensable, to change a complex number from the orthogonal to the exponential or the trigonometric form and vice versa. Such a change involves the operations of computing the modulus and argument by the use of Eqs. (6) and (7). The author has developed an instrument which accomplishes this result in one setting. The use of this instrument, in connection with this text, is recommended.

The subtraction of complex numbers is performed by separately subtracting the real- and the j -terms. Thus

$$(a + ja') - (b + jb') = (a - b) + j(a' - b') \quad (27)$$

Complex numbers can be actually added or subtracted only when expressed in the orthogonal form. Hence a complex number, expressed in any form other than the orthogonal, must be changed to the orthogonal form before proceeding with the operations of addition or subtraction.

8. Multiplication of Complex Numbers. — Unlike addition and subtraction, the multiplication of complex numbers may be accomplished irrespective of the form in which these numbers are expressed. We shall show how this operation is done when complex numbers are expressed in the orthogonal, exponential, and circular forms:

(a) *Orthogonal Form.* — The multiplication of two complex numbers expressed in the orthogonal form follows the algebraic rules for multiplication where $j = \sqrt{-1}$. Thus:

$$\begin{aligned} (a + ja')(b + jb') &= ab + j(a'b + ab') + j^2a'b' \\ &= (ab - a'b') + j(a'b + ab') \end{aligned} \quad (28)$$

(b) *Exponential Form.* — Let the complex numbers **A** and **B** be expressed in the exponential form. The product **P** is:

$$\mathbf{P} = A\epsilon^{j\alpha} B\epsilon^{j\beta} = AB\epsilon^{j(\alpha+\beta)} \quad (29)$$

From Eq. (29) the following general rule for multiplying complex numbers may be deduced:

The product **P** of a set of complex numbers **A**, **B**, **C**, . . . **N** is a complex number having a modulus equal to the product of the moduli and an argument equal to the sum of the arguments of its factors. Thus:

$$\mathbf{P} = P\epsilon^{j\phi} = (ABC \dots N)\epsilon^{j(\alpha+\beta+\gamma+\dots+\omega)} \quad (30)$$

where:

$$\begin{aligned} P &= (ABC \dots N) & (a) \} \\ \phi &= (\alpha + \beta + \gamma + \dots + \omega) & (b) \} \end{aligned} \quad (31)$$

(c) *Circular Form.* — Let the complex numbers **A** and **B** be expressed in the circular form. The product **P** is:

$$\begin{aligned} \mathbf{P} &= A(\cos \alpha + j \sin \alpha) B(\cos \beta + j \sin \beta) \\ &= AB[(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + j(\sin \alpha \cos \beta + \cos \alpha \sin \beta)] \\ &= AB[\cos(\alpha + \beta) + j \sin(\alpha + \beta)] = AB \operatorname{cis}(\alpha + \beta) = AB \angle(\alpha + \beta) \end{aligned} \quad (32)$$

The identity of Eqs. (29) and (32) is at once apparent (see Eq. 20).

9. Division of Complex Numbers. — Like multiplication, the division of complex numbers may be accomplished when these numbers are expressed in the orthogonal, exponential, or circular forms.

(a) *Orthogonal Form.* — The division of two complex numbers expressed in the orthogonal form is accomplished by multiplying the numerator and denominator by the conjugate* of the denominator and simplifying. Thus:

$$\begin{aligned} Q &= (a + ja')/(b + jb') \\ &= (a + ja') (b - jb')/(b + jb') (b - jb') \\ &= [ab + a'b' + j(a'b - ab')]/(b^2 + b'^2) \end{aligned} \quad (33)$$

(b) *Exponential Form.* — Let the two complex numbers **A** and **B** be expressed in the exponential form. Then the quotient is:

$$Q = (Ae^{j\alpha}/Be^{j\beta}) = (A/B)e^{j(\alpha-\beta)} = Qe^{j\theta} \quad (34)$$

where:

$$\left. \begin{aligned} Q &= A/B & (a) \\ \theta &= \alpha - \beta & (b) \end{aligned} \right\} \quad (35)$$

Hence the quotient **Q**, obtained by the division of a complex number **A** by another complex number **B**, is a complex number having a modulus equal to the quotient of the moduli and an argument equal to the difference of the arguments of the dividend and divisor.

(c) *Trigonometric or Circular Form.* — Let the two complex numbers be expressed in the circular form. Then the quotient is:

$$\begin{aligned} Q &= \frac{A(\cos \alpha + j \sin \alpha)}{B(\cos \beta + j \sin \beta)} \\ &= \frac{A(\cos \alpha + j \sin \alpha) (\cos \beta - j \sin \beta)}{B(\cos \beta + j \sin \beta) (\cos \beta - j \sin \beta)} \\ &= (A/B)[(\cos \alpha \cos \beta + \sin \alpha \sin \beta) + j(\sin \alpha \cos \beta - \cos \alpha \sin \beta)] \\ &= (A/B)[\cos(\alpha - \beta) + j \sin(\alpha - \beta)] = (A/B) \operatorname{cis}(\alpha - \beta) \\ &= (A/B) \angle (\alpha - \beta) \\ &= Q (\cos \theta + j \sin \theta) \end{aligned} \quad (36)$$

The identity of Eqs. (34) and (36) is proved in Art. 5d above.

10. Powers and Roots of Complex Numbers. — Let it be required to raise a complex number to the *n*th power. We have:

$$A^n = (Ae^{j\alpha})^n = A^n e^{jn\alpha} \quad (37)$$

* Two complex numbers are said to be conjugate when they have equal real terms of the same sign, and equal *j*-terms of opposite signs. Thus the numbers $b + jb'$ and $b - jb'$ are conjugate.

Now n , in Eq. (37), may be real, imaginary, or complex. Consider the general case where n is a complex number. Let:

$$n = u + jv \quad (38)$$

Substituting Eq. (38) in (37) we have:

$$\begin{aligned} \mathbf{A}^{(u+jv)} &= (A\epsilon^{j\alpha})^{(u+jv)} = A^{(u+jv)}\epsilon^{j(u+jv)\alpha} \quad (a) \\ &= A^u A^{jv}\epsilon^{ju\alpha}\epsilon^{-v\alpha} \quad (b) \end{aligned} \quad (39)$$

By definition of a logarithm we have:

$$A^{jv} = \epsilon^{jv \ln A} \quad (40)$$

Substituting Eq. (40) in Eq. (39b) we obtain:

$$\begin{aligned} \mathbf{A}^{(u+jv)} &= A^u \epsilon^{-v\alpha} \epsilon^{ju\alpha} \epsilon^{jv \ln A} \quad (a) \\ &= A^u \epsilon^{-v\alpha} [\epsilon^{j(u\alpha + v \ln A)}] \quad (b) \\ &= A^u \epsilon^{-v\alpha} [\cos(u\alpha + v \ln A) + j \sin(u\alpha + v \ln A)] \quad (c) \end{aligned} \quad (41)$$

Eqs. (41) are the most general expressions for a complex number raised to any power — real, imaginary, or complex. Thus if a complex number is raised to a real power, then v in Eqs. (41) is zero and we have:

$$\mathbf{A}^u = A^u \epsilon^{ju\alpha} = A^u (\cos u\alpha + j \sin u\alpha) \quad (41')$$

On the other hand if a complex number is raised to an imaginary power, then u in Eqs. (41) is zero and we have:

$$\mathbf{A}^{jv} = \epsilon^{-v\alpha} \epsilon^{jv \ln A} = \epsilon^{-v\alpha} [\cos(v \ln A) + j \sin(v \ln A)] \quad (41'')$$

From Eqs. (41) we conclude that the result of raising a complex number $\mathbf{A}$ to a power $u + jv$ is a complex number whose modulus is $A^u \epsilon^{-v\alpha}$ and whose argument is $(u\alpha + v \ln A)$. *This argument must be always expressed in radians.*

11. Effect of the Periodicity of the Argument on the Powers and Roots of a Complex Number. — An angle repeats itself every 2π radians. Hence:—

$$A\epsilon^{j\alpha} = A\epsilon^{j(\alpha \pm 2n\pi)} \quad (42)$$

The two arguments α and $(\alpha \pm 2n\pi)$ can be used interchangeably in all the algebraic operations, defined in Arts. 6 to 10 above, except the following:

- (a) When a complex number is raised to a real power u which is not an integer.
- (b) When a complex number is raised to an imaginary power jv .
- (c) When a complex number is raised to a complex power $u \pm jv$.

In each of these operations the results will be different as various multiples of 2π are added to the argument. Mathematically all these values are possible. In most physical problems, however, only one of these values actually satisfies the physical facts and must, therefore, be selected as the value desired.

12. Solution of Simultaneous Equations Involving Complex Numbers.

— Let it be required to solve the equations:

$$\left. \begin{aligned} \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{Y} &= \mathbf{C} & (a) \\ \mathbf{D}\mathbf{X} + \mathbf{E}\mathbf{Y} &= \mathbf{F} & (b) \end{aligned} \right\} \quad (43)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{F}$ are known, $\mathbf{X}$, and $\mathbf{Y}$ are unknown complex numbers.

The process of solving these equations for $\mathbf{X}$ and $\mathbf{Y}$ does not differ at all from the ordinary method used with equations involving real numbers. Thus, multiplying Eq. (43a) by $\mathbf{D}$ and Eq. (43b) by $\mathbf{A}$ subtracting and solving for $\mathbf{Y}$ we obtain:

$$\mathbf{Y} = (\mathbf{CD} - \mathbf{AF})/(\mathbf{BD} - \mathbf{AE}) \quad (44)$$

Similarly eliminating $\mathbf{Y}$ from Eqs. (43) and solving for $\mathbf{X}$ we have:

$$\mathbf{X} = (\mathbf{CE} - \mathbf{BF})/(\mathbf{AE} - \mathbf{BD}) \quad (45)$$

13. The Imaginary Unit (j) as a Rotating Operator. — Let a real number a be multiplied by j once, twice, thrice, and four times. This gives respectively:

$$\left. \begin{aligned} j(a) &= ja & (a) \\ j \cdot j \cdot (a) &= j^2a = -a & (b) \\ j \cdot j \cdot j \cdot (a) &= -ja & (c) \\ j \cdot j \cdot j \cdot j \cdot (a) &= a & (d) \end{aligned} \right\} \quad (46)$$

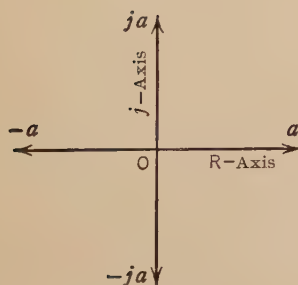


Fig. 3a

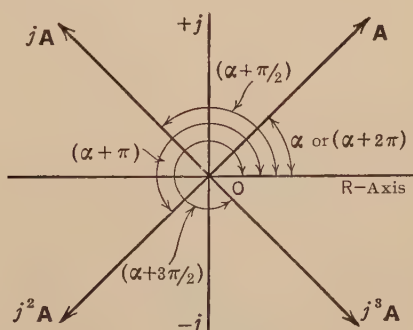


Fig. 3b

FIG. 3. The imaginary unit (j) as a rotating operator.

Referring to Fig. 3a it will be observed that each number in Eqs. (46) is displaced by $\pi/2$ radians counterclockwise from the number preceding

it. We thus conclude that each time a number (real or imaginary) is multiplied by j , this number is rotated by $\pi/2$ radians counterclockwise.

The foregoing deduction applies to real- and j -numbers. To show that a multiplication by j rotates a complex number by $\pi/2$ radians let j be expressed in the following form:

$$j = \epsilon^{j\pi/2} \quad (47)$$

Now let a complex number $A\epsilon^{j\alpha}$ be multiplied by j once, twice, thrice, and four times. The products are respectively:

$$\left. \begin{aligned} jA\epsilon^{j\alpha} &= \epsilon^{j\pi/2}A\epsilon^{j\alpha} = A\epsilon^{j(\alpha+\pi/2)} & (a) \\ j \cdot j \cdot A\epsilon^{j\alpha} &= \epsilon^{j\pi}A\epsilon^{j\alpha} = A\epsilon^{j(\alpha+\pi)} & (b) \\ j \cdot j \cdot j \cdot A\epsilon^{j\alpha} &= \epsilon^{j3\pi/2}A\epsilon^{j\alpha} = A\epsilon^{j(\alpha+3\pi/2)} & (c) \\ j \cdot j \cdot j \cdot j \cdot A\epsilon^{j\alpha} &= \epsilon^{j2\pi}A\epsilon^{j\alpha} = A\epsilon^{j(\alpha+2\pi)} & (d) \end{aligned} \right\} \quad (48)$$

Eqs. (48) show that each multiplication by j increases the argument of a complex number by $\pi/2$ radians which means a rotation by $\pi/2$ radians counterclockwise. Thus let the complex number $A\epsilon^{j\alpha}$ be represented graphically through its modulus and argument, then the complex numbers (Eqs. 48) may be represented by the four vectors shown in Fig. 3b. These four vectors are displaced from each other by $\pi/2$ radians.

14. The Rotating Operator j^n . — Since a vector is rotated counterclockwise by $\pi/2$ radians when multiplied by j^1 it is natural to ask whether j^n (n any real number) may be adopted as an operator to rotate a complex number by any desired angle. That this is possible may be shown by the following reasoning:

If a multiplication by j^1 rotates a complex number by $\pi/2$ radians then a multiplication by j^n should cause a rotation by $n\pi/2$ radians because the operator is applied n times. Hence in order to rotate a complex number by an angle β it is necessary:

- (a) To determine the number of times, n , that the angle $\pi/2$ is contained in β .
- (b) To apply the operator, j , n — times or multiply the number by j^n .

But the number of times, n , that the angle $\pi/2$ is contained in β is obtained by simply dividing β by $\pi/2$. This gives:

$$n = 2\beta/\pi \quad (49)$$

Hence if a complex number is multiplied by $j^{2\beta/\pi}$, that number will be rotated by β radians. This deduction can be proved mathematically

by making use of Eq. (47). We have:

$$j^{2\beta/\pi} A \epsilon^{j\alpha} = (\epsilon^{j\pi/2})^{2\beta/\pi} A \epsilon^{j\alpha} = \epsilon^{j\beta} A \epsilon^{j\alpha} = A \epsilon^{j(\alpha+\beta)} \quad (50)$$

Eq. (50) shows that the argument of the complex number $A \epsilon^{j\alpha}$ is increased by β radians as a result of multiplying that complex number by $j^{2\beta/\pi}$. In other words the complex number has been rotated by β radians counterclockwise.

Although it may be sometimes convenient to use the rotating operator j^n in the form $j^{2\beta/\pi}$ it is often more convenient to use it in the form $\epsilon^{j\beta}$. The identity of the two forms may be established by the use of Eq. (47). Thus:

$$j^{2\beta/\pi} = (\epsilon^{j\pi/2})^{2\beta/\pi} = \epsilon^{j\beta} \quad (51)$$

PROBLEMS

1. Give a few equations which yield imaginary roots, and solve for the roots.
2. Evaluate j^5 , j^9 , j^{12} , j^{45} , j^{72} , j^{115} .
- 3a. Give two equations which yield complex roots and determine the roots.
- 3b. A vector voltage $\mathbf{E}$ has a real component of 50 volts and a j -component of 80 volts. Give the expression for the voltage in the complex form.
4. Plot the following complex numbers in the R - j plane:
 $2 + j3$; $4 - j2$; $-5 + j6$; $-3 - j4$; $0 + j2$; $0 - j4$; $4 + j0$; $2 - j0$.
- 5a. Change the following complex numbers to the exponential and circular forms:
 (a) $756 + j875$; (b) $1157 - j976$; (c) $-2315 + j154$; (d) $-9720 - j21$;
 (e) $0.25 + j3.75$; (f) $-0.0532 - j2.85$; (g) $-71 + j32.1$.
- 5b. Express the following complex numbers in the orthogonal form:
 (a) $126 \angle 55^\circ 20'$; (b) $351 \angle 20^\circ 15'$; (c) $-2.432 \angle 195^\circ 25'$;
 (d) $-3.345 \angle 217^\circ 10'$.
- 5c. Evaluate: $\epsilon^{j\pi/2}$; $\epsilon^{-j\pi/2}$; $\epsilon^{j\pi}$; $\epsilon^{j2\pi}$
- 5d. Show that $\sin \alpha = (\epsilon^{j\alpha} - \epsilon^{-j\alpha})/2j$.
- 5e. Show that $\cos \beta = (\epsilon^{j\beta} + \epsilon^{-j\beta})/2$.
- 5f. Show by Maclaurin's Theorem that:
 $A \epsilon^{j\alpha} = A(\cos \alpha + j \sin \alpha)$.
- 6a. Given: $x + jy = 3 + j4$. Determine x and y .
- 6b. Given: $(\sqrt{x^2 + y^2}) \epsilon^{j\beta} = 5 + j6$. Determine x , y and β .
- 7a. Add the following complex numbers:
 (a) $3 + j8$; $9 - j6$; $7 + j5$.
 (b) $8 \angle 30^\circ$; $9 \epsilon^{j\pi/2}$; $10 \text{ cis } 45^\circ$.
- 7b. Subtract $7 + j5$ from $9 - j6$.
- 7c. Subtract $15 \text{ cis } 30^\circ$ from $5 \text{ cis } 25^\circ$.

8a. Evaluate:

- (a) $(10 + j 21.4) (-5 - j 17.2) (6 - j 114) (-712 + j 2.15)$.
 (b) $(12.72 + j 125) (-13 + j 25) (-12 - j 125)$.
 (c) $(11 - j 23) (-12 - j 175)$.

8b. Change each of the complex numbers given in problem 8a to the exponential or circular form and perform the required products. Check your results with those obtained in problem 8a.

9a. Evaluate:

- (a) $(10 + j 213) (-712 + j 22) / (-6 - j 171) (600 - j 1120)$
 (b) $(12.8 + j 125) / (13 + j 27) (-12 - j 125)$
 (c) $(11 - j 23) (-12 - j 175) / (12.8 + j 126)$

9b. Change each of the complex numbers given in problem 9a to the exponential or circular form and perform the divisions required then check your results with those obtained in problem 9a.

10a. Evaluate:

- (a) $(11.60 - j 9.8)^{2.57}$; (b) $(-9.8 - j 21)^{1.34}$.
 (c) $(-23.2 + j 154)^{0.36}$; (d) $(26 + j 4)^{-2.8}$.

10b. Evaluate $(2 + j 5)^{j 7}$.

10c. Evaluate $(3 + j 4)^{(5 + j 8)}$.

10d. Given: $(a + jb)^c = g + jh$. Express g and h in terms of a , b , and c .

10e. Given: $(a + jb)^{jd} = k + jl$. Express k and l in terms of a , b , and d .

10f. Given: $(a + jb)^{(c + jd)} = m + jn$. Express m and n in terms of a , b , c , and d .

10g. Prove by the use of complex numbers that:

$$\left. \begin{aligned} \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta & (a) \\ \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta & (b) \end{aligned} \right\} \quad (52)$$

10h. Show that $e^{j(\alpha + \pi/2)} = j e^{j\alpha}$ and that $e^{j(\alpha - \pi/2)} = -j e^{j\alpha}$.

10i. Evaluate: (a) 1^j ; (b) j^j .

11a. Show that the solution of the equation $x^3 - 1 = 0$ gives three roots of x one of which is real and the other two are imaginary. (*Hint*: replace 1 by $e^{j2n\pi}$.)

11b. Show that, in general, the solution of $x^n - 1 = 0$ gives n roots of x such that $x = e^{jp(2\pi/n)}$ where $p = 0, 1, 2, 3, \dots, n$.

11c. Show that $(A e^{j\alpha})^{1/3}$ has three distinct values. Give these values.

11d. Show that $(A e^{j\alpha})^{jb}$ has an infinite number of values corresponding to the values of n taken when $A e^{j\alpha}$ is replaced by $A e^{j(\alpha \pm 2n\pi)}$.

11e. Show that $(A e^{j\alpha})^{(a + jb)}$ has an infinite number of values corresponding to the values of n taken when $A e^{j\alpha}$ is replaced by $A e^{j(\alpha \pm 2n\pi)}$.

12a. The equations expressing Kirchhoff's laws for an electric circuit are:

$$\begin{aligned} \mathbf{E} - \mathbf{I}_1 \mathbf{Z}_1 - \mathbf{I}_2 \mathbf{Z}_2 &= 0 \\ \mathbf{I}_2 \mathbf{Z}_2 + \mathbf{I}_3 \mathbf{Z}_3 &= 0 \\ \mathbf{I}_1 - \mathbf{I}_2 - \mathbf{I}_3 &= 0 \end{aligned}$$

Assuming that $\mathbf{E}$, $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ are known, what are the values of $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.

12b. Let the known quantities in problem 12a be:

$$\mathbf{E} = 100 + j\,50$$

$$\mathbf{Z}_1 = 5 + j\,7$$

$$\mathbf{Z}_2 = 8 + j\,12$$

$$\mathbf{Z}_3 = 10 + j\,15$$

Compute the values of $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.

13a. Show that a multiplication by j^π rotates a complex number counterclockwise by $\pi/2$ radians and is therefore equivalent to a multiplication by j .

13b. Show that when a complex number is divided by j it will be rotated by $\pi/2$ radians clockwise.

14a. Evaluate $j^{\pi/4}(5 + j\,8)$.

14b. Express the numbers given in problem 5c as powers of j (for example $e^{j\pi/2} = j^1$).

CHAPTER II

VECTORS

1. Definitions. — The following terms are used throughout the text; hence it is necessary to define them:

(a) *A Vector* is a quantity possessing both a magnitude and a direction. Thus force, magnetic flux density, and potential gradient are vectors. These physical quantities are completely defined only when their magnitudes and their directions are given.

(b) *A Scalar* is a quantity possessing magnitude but no direction. Mass, power, and energy are examples of scalar quantities.

(c) *Origin and Terminus of a Vector.* — Every vector has an origin and a terminus. The origin of a vector is the point whence it starts and its terminus is its end point. The terminus of a vector is indicated by an arrow. Thus, the vector **A** (Fig. 1) originates at the point *O* and terminates at the point indicated by the arrow.

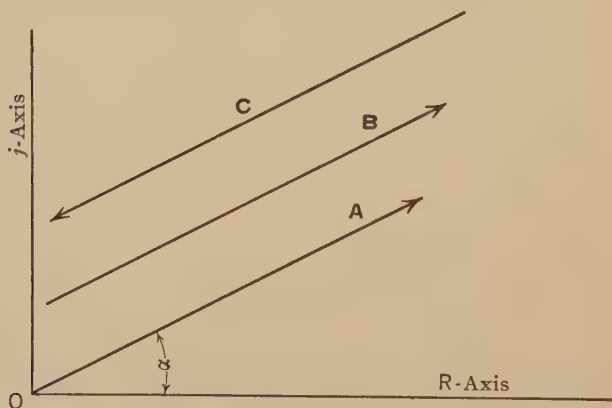


FIG. 1. Equal vectors and negative vectors.

(d) *Coplanar Vectors.* — Any two or more vectors which lie in the same plane are coplanar. We shall deal exclusively with coplanar vectors lying in the *R-j* plane. The vectors **A**, **B**, and **C** (Fig. 1) are coplanar.

(e) *Equal or Parallel Vectors.* — Two vectors are equal when they have the same magnitude and the same direction. The vectors **A** and **B** (Fig. 1) are equal.

(f) *Negative or Anti-Parallel Vectors.* — Two vectors are the negative of each other when they are of the same magnitude but are oppositely directed. The Vectors **B** and **C** (Fig. 1) are mutually negative.

(g) *Conjugate Vectors.* — Two coplanar vectors, lying in the R - j plane, are mutually conjugate when they have equal projections along the R -axis but opposite projections along the j -axis. In Fig. 2, the vector $\mathbf{A}_1$ is the conjugate of $\mathbf{A}_2$ and the vector $\mathbf{B}_1$ is the conjugate of $\mathbf{B}_2$. The fact that two vectors are mutually conjugate is indicated by placing a dash on the symbol representing them. Thus the vectors **A** and $\bar{\mathbf{A}}$ are mutually conjugate.*

(h) *Unit Vector.* — A unit vector ($\mathbf{a}_1$ Fig. 2) is a vector having unit length in the direction of **A**.

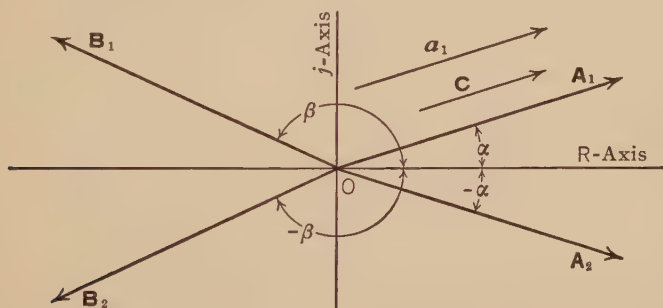


FIG. 2. Conjugate vectors, unit vector and reciprocal vectors.

(i) *Reciprocal Vectors.* — Two vectors are mutually reciprocal when they have the same direction but reciprocal magnitudes. Thus the vectors **C** and $\mathbf{A}_1$, Fig. 2, are mutually reciprocal because they have the same direction but their magnitudes are so related that the magnitude of **C** is $0.5 \mathbf{a}_1$ while the magnitude of $\mathbf{A}_1$ is $2 \mathbf{a}_1$.

2. The Use of Complex Numbers to Represent Vectors in the R - j -Plane. — A vector **A** (Fig. 1) lying in the R - j -plane is completely defined through its length A and the angle α it makes with the R -axis. But the *length* of a vector corresponds to the modulus of a complex number and its *direction-angle*, α , corresponds to the argument of a complex number. Hence complex numbers may be used to define vectors in the R - j -plane. Thus the vector **A** (Fig. 1) may be defined as follows:

$$\mathbf{A} = A e^{j\alpha} = A(\cos \alpha + j \sin \alpha) = a + ja' \quad (1)$$

where A is the magnitude of the vector or the modulus of the complex number, and α the direction-angle of the vector or the argument of the complex number representing that vector.

* See definition of conjugate complex numbers given in footnote, page 7.

Naturally the vector $\mathbf{A}$ may be also defined through its projections along the R - and j -axes. These are: $A \cos \alpha = a$, and $A \sin \alpha = a'$. a and a' correspond respectively to the real- and j -parts of the complex number representing the vector $\mathbf{A}$.

Although complex numbers may be used to define vectors in the R - j -plane they do not only represent vectors. Indeed these numbers are used, in electrical engineering, to represent not only vectors but also *operators* which are scalar quantities. When a complex number represents a scalar operator, the arrow will be omitted (see Fig. 2, Chapter I). This distinction between operators and vectors is also rigidly observed in the notations by using bold-face type for vectors and regular type for operators. Thus the letter $\mathbf{I}$ represents a vector whereas the letter Z represents an operator.

3. Addition of Vectors. — Vectors may be added either graphically or analytically.

(a) To add vectors graphically, join their equals in such a manner that the origin of one vector is at the terminus of the preceding one. Then their sum is represented by the vector whose origin coincides with the origin of the first and whose terminus is that of the last vector. Thus in Fig. 3:

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 \quad (2)$$

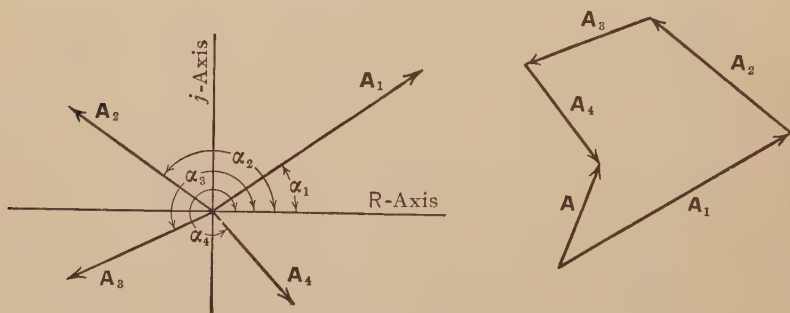


FIG. 3. Graphic addition of vectors.

(b) The analytical addition of coplanar vectors is accomplished by first expressing them in the complex form and then proceeding as in the addition of complex numbers. Thus let the vectors shown in Fig. 3 have moduli A_1, A_2, A_3, A_4 and direction-angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. These vectors can be expressed in the orthogonal form as follows:

$$\left. \begin{aligned} \mathbf{A}_1 &= A_1(\cos \alpha_1 + j \sin \alpha_1) = a_1 + ja'_1 & (a) \\ \mathbf{A}_2 &= A_2(\cos \alpha_2 + j \sin \alpha_2) = a_2 + ja'_2 & (b) \\ \mathbf{A}_3 &= A_3(\cos \alpha_3 + j \sin \alpha_3) = a_3 + ja'_3 & (c) \\ \mathbf{A}_4 &= A_4(\cos \alpha_4 + j \sin \alpha_4) = a_4 + ja'_4 & (d) \end{aligned} \right\} \quad (3)$$

Then their sum is:

$$\mathbf{A} = (a_1 + a_2 + a_3 + a_4) + j(a'_1 + a'_2 + a'_3 + a'_4) \quad (4)$$

4. Subtraction of Vectors. — Vectors may be subtracted either graphically or analytically.

(a) To subtract vectors graphically first draw the negatives of the vectors to be subtracted, then proceed as in addition. Thus in Fig. 4:

$$\mathbf{A} = \mathbf{A}_1 - \mathbf{A}_2 - \mathbf{A}_3 \quad (5)$$

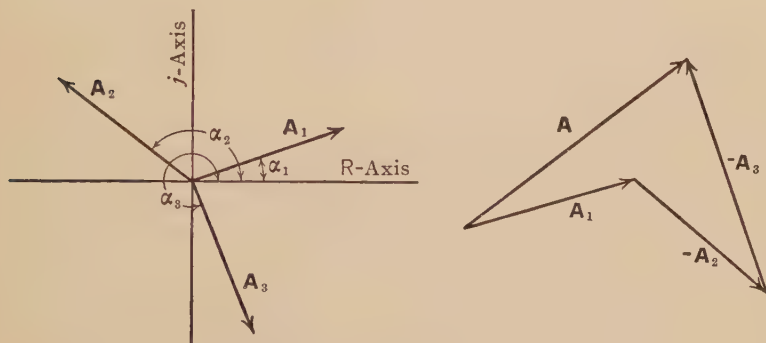


FIG. 4. Graphic subtraction of vectors.

(b) The analytical subtraction of coplanar vectors is accomplished by (1) expressing them in the complex form, (2) multiplying both the real- and j -terms of the vectors to be subtracted by -1 , and (3) proceeding as in the addition of complex numbers. Thus:

$$\mathbf{A} = \mathbf{A}_1 - \mathbf{A}_2 - \mathbf{A}_3 = (a_1 - a_2 - a_3) + j(a'_1 - a'_2 - a'_3) \quad (6)$$

The order in which vectors are added or subtracted does not affect the result.

5. Multiplication of Vectors. — We shall define four distinct types of products each of which finds practical application in electrical engineering:

(a) *The Product of a Vector by a Scalar.* — When a vector ($\mathbf{A}$) is multiplied by a scalar positive number (b), a vector is obtained having a modulus equal to bA , and an argument equal to that of the original vector. If the scalar multiplier is a negative number, then a negative vector (see Art. 1f) is obtained. If the scalar multiplier is zero, then the vector vanishes.

(b) *The Complex Product of Vectors.* — The complex product of two or more coplanar vectors or of vectors and operators is a coplanar vector having a modulus equal to the product of the moduli and an argument equal to the sum of the arguments of its factors. We shall adopt the

notation $\mathbf{AB}$ to mean the complex product of $\mathbf{A}$ and $\mathbf{B}$. This definition is identical with that given for the multiplication of complex numbers (see Art. 8, Chapter I). The complex product of two or more coplanar vectors may be obtained by first expressing these vectors in the complex form and then proceeding to multiply them as indicated in Art. 8, Chapter I.

(c) *The Scalar or Dot Product of Two Vectors.* — The scalar or dot product of two vectors is a scalar of magnitude equal to the product of the moduli of the two vectors multiplied by the cosine of the angle between them. The notation generally adopted for the dot product is a dot between the two vectors. Thus:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta = AB \cos (\alpha - \beta) \quad (7)$$

where α and β are the arguments of $\mathbf{A}$ and $\mathbf{B}$ respectively.

The reader will at once perceive that Eq. (7) represents the definition of average electric power if A is the effective value of the sine voltage and B is the effective value of the sine current in an electric circuit.

(d) *The Cross or Vector Product of Two Vectors.* — The cross or vector product of two vectors is a vector whose modulus is equal to the product of the moduli of the two vectors multiplied by the sine of the angle between them. Moreover, the product $\mathbf{P}$ (see Fig. 5) is directed along the perpendicular to the plane of the two vectors such that when the first vector is rotated toward the second, the product points in the direction of the progress of a right-hand screw similarly rotated. The notation

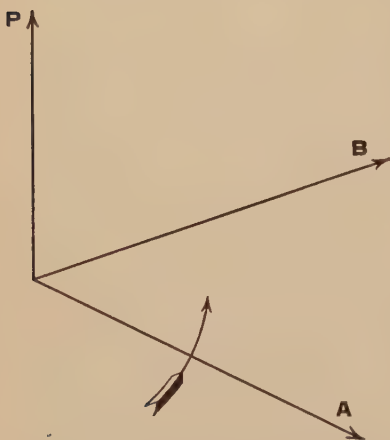


FIG. 5. The cross product $\mathbf{P} = \mathbf{A} \times \mathbf{B}$.

adopted for this type of product is a cross ($\times$) between the two vectors. Thus:

$$\mathbf{A} \times \mathbf{B} = \mathbf{P} = AB \sin (\theta) \mathbf{p}_1 \quad (8)$$

where $\mathbf{p}_1$ is a unit vector perpendicular to $\mathbf{A}$ and $\mathbf{B}$ and properly directed so as to satisfy the definition of the cross product (see Fig. 5) and $\theta = (\alpha - \beta)$ is the angle between $\mathbf{A}$ and $\mathbf{B}$. By definition we have:

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} \quad (9)$$

The reason for the negative sign is that by definition the direction of $\mathbf{p}_1$ (Eq. 8) is along the progress of a screw rotated from the first vector toward the second. If the order of the vectors is reversed so that $\mathbf{B}$ is the first and $\mathbf{A}$ the second vector appearing in the cross product, then naturally the direction of rotation is reversed and the direction of the progress of the screw is also reversed.

Eqs. (8) and (9) will undoubtedly recall to the reader's mind the relation between flux, current and force — the right-hand rule, so familiar to engineers.

6. Division. — There exists but one type of vector division (the complex quotient) which is of practical interest in this text. This will be defined as follows:

The *Complex quotient*, obtained by dividing a vector by an operator or a vector by another vector is respectively a vector or an operator having a modulus equal to the quotient of the moduli and an argument equal to the difference of their arguments. This definition is identical with that given for the division of complex numbers (see Art. 9, Chapter I). The complex quotient may therefore be obtained by first expressing the vectors and the operator in the complex form and then proceeding with the division as indicated in Art. 9, Chapter I.

PROBLEMS

1a. Give examples of vector and scalar quantities other than those cited in the text.

1b. Draw a vector $\mathbf{B}$, and indicate its origin and terminus.

1c. Draw (a) two equal vectors; (b) two negative vectors; (c) two conjugate vectors; (d) four coplanar vectors; (e) two reciprocal vectors.

2a. A vector $\mathbf{A}$, lying in the R - j plane, has a modulus of 5 cm. and a direction-angle of 30° .

(a) Plot the vector on coördinate paper.

(b) Express it in the exponential and orthogonal complex forms.

(c) Draw $-\mathbf{A}$; $\bar{\mathbf{A}}$; a_1 .

3a. Four coplanar vectors have moduli $A = 5$; $B = 6$; $C = 7$; $D = 8$; and direction-angles $\alpha_1 = 20^\circ$; $\alpha_2 = 70^\circ$; $\alpha_3 = 40^\circ$ and $\alpha_4 = 180^\circ$.

Determine both graphically and analytically:

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}.$$

$$\bar{\mathbf{A}} + \bar{\mathbf{B}} + \bar{\mathbf{C}} + \bar{\mathbf{D}}.$$

3b. Three coplanar vectors have equal moduli and are displaced from each other by 120° . Show both graphically and analytically that their sum is zero.

3c. Show that the sum of ($n \geq 2$) coplanar vectors, of equal moduli and equally displaced from each other by an angle $\alpha = 2\pi/n$ degrees, is zero.

NOTE. — The solution of this problem by the use of complex numbers is given in Appendix I.

3d. Show graphically and analytically that the sum of any number of vectors which form a closed polygon (regular or irregular) is zero.

4. Referring to the vectors given in problem 3a, determine graphically and analytically:

$$(a) \mathbf{A} - \mathbf{B} + \mathbf{C} - \mathbf{D}.$$

$$(b) \mathbf{A} - \mathbf{B} - \mathbf{C} + \mathbf{D}.$$

$$(c) \mathbf{A} - \bar{\mathbf{B}} + \mathbf{C} - \bar{\mathbf{D}}.$$

$$(d) \bar{\mathbf{A}} - \bar{\mathbf{B}} - \bar{\mathbf{C}} + \bar{\mathbf{D}}.$$

5a. Multiply the vector $\mathbf{A}$ given in problem 3a by 2, 0.5, 0, -2 , and -0.5 respectively. Then (a) plot the results on coördinate paper, and (b) give the complex expression for each of the vectors so obtained.

5b. Determine the following complex products of the vectors:

$$(a) \mathbf{E} \mathbf{I}$$

$$(b) \mathbf{E} \bar{\mathbf{I}}$$

where

$$\mathbf{I} = 10 + j 2 \quad \text{and} \quad \mathbf{E} = 90 + j 60.$$

5c. Evaluate the dot product of the vectors given in problem 5b.

5d. Evaluate the cross product ($\mathbf{E} \times \mathbf{I}$) of the vectors specified in problem 5b.

6a. Find the complex quotient ($\mathbf{E}/\mathbf{I}$) of the vectors specified in problem 5b.

CHAPTER III

SINE FUNCTIONS*

1. **Definition.** — A sine function is of the form:

$$e = f(x) = E_m \sin (x + \alpha) \quad (1)$$

In Eq. (1) E_m and α are constants. The dependent variable e is a function of the argument x . In other words, to every arbitrarily assigned value of x there corresponds a certain definite value of e . Thus let the argument x and the constant α be expressed in degrees or radians. Then, for any value of x , the value of the expression $\sin (x + \alpha)$ may be computed by the use of Table I, Appendix II. Multiplying this by E_m we obtain e .

2. **Graphic Representation of Sine Functions.** — Graphs of the sine function given in Eq. (1) may be easily obtained by the use of the rectangular or the polar system of coördinates. If the curve be plotted

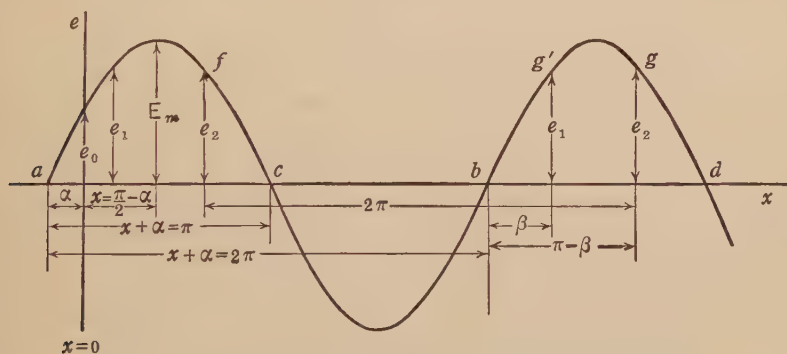


FIG. 1. A sine function plotted in rectangular coördinates.

in rectangular coördinates, then x is taken as abscissa and e as ordinate. This gives a curve similar to that shown in Fig. 1. Such a curve is called a sine wave. The same function plotted in polar coördinates, over a range of 180° , is shown in Fig. 2. Here the argument $(x + \alpha)$ is plotted as the vectorial angle and the dependent variable e is taken as the radius

* This chapter is devoted to the study of the sines of the variable x and not of the multiples of x (for example, $\sin 2x$, $\sin 3x$ etc.). The study of this type of functions is reserved to Chapter XIII.

vector. The general practice favors the rectangular system. Accordingly all sine waves in the text are plotted in this system.

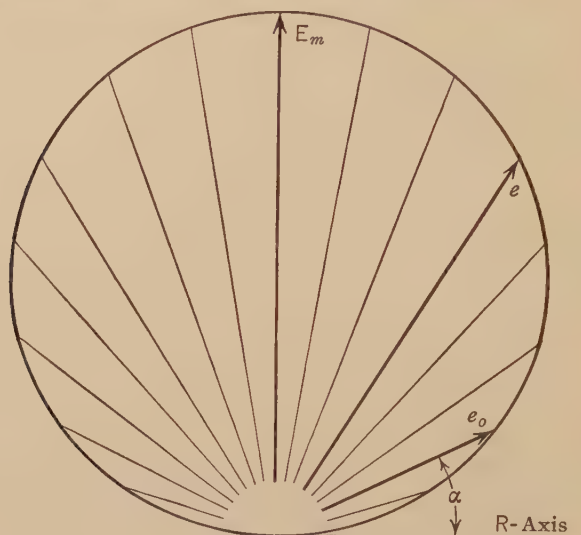


FIG. 2. A sine function plotted in polar coördinates.

3. Graphic Meaning of the Symbols Used in Eq. (1). — The symbols used in Eq. (1) will now be defined and interpreted graphically.

(a) *Instantaneous Value e .* — The dependent variable e (Eq. 1) shall be termed the *instantaneous value* of the sine function. Graphically e represents the ordinate of any point on the curve Fig. 1.

Four particular instantaneous values of a sine function deserve special mention. These are (1) the amplitude; (2) the initial value; (3) the zero values; and (4) the π values.

(1) *The Amplitude E_m .* — A glance at Fig. 1 shows that there exists one ordinate which is the largest. This ordinate is known as the *amplitude* or *maximum instantaneous value* or merely *maximum value* of a sine function. It is denoted by the symbol E_m . The amplitude occurs when $\sin(x + \alpha) = 1$.

(2) *The Initial Value e_0 .* — The initial value of a sine function is that instantaneous value which corresponds to the angle $x = 0$. Thus substituting $x = 0$ in Eq. (1) we have

$$e_0 = E_m \sin \alpha \quad (2)$$

The initial value e_0 is shown on the graph (Fig. 1) as the ordinate corresponding to the abscissa $x = 0$.

(3) *The Zero-Values.* — The zero-values of a sine function are the instantaneous values which correspond to the angle $(x + \alpha) = 0$ or $2n\pi$ (where n is any integer). Graphically the zero-values of a sine function are points on the sine wave whose ordinates are zero. Moreover the slope of the curve at these points is positive. The points a and b , Fig. 1, represent two zero-values of the sine function.

(4) *The π -Values.* — The π -values of a sine function are those instantaneous values which correspond to the angle $(x + \alpha) = \pi$. Graphically the π -values represent points on the sine wave whose ordinates are zero. Moreover the slope of the curve at those points is negative. The points c and d , Fig. 1, are two π -values of the sine function.

The distinction drawn between the zero- and the π -values of a sine function must be clearly understood. The ordinates in either case are zero. But the tangent to the curve is positive at the zero-points and negative at π -points.

(b) *The Argument x .* — The argument x (Eq. 1) is a variable angle to every value of which there corresponds one and only one instantaneous value e . Consequently a sine function is a single-valued function. Graphically the argument x represents the distance measured along the abscissa, to the right of an arbitrary point $x = 0$ (see Fig. 1).

(c) *The Initial Angle α .* — The constant angle α (Eq. 1) is called the initial angle because its value determines the initial value e_0 of a sine function. Graphically α represents the distance (Fig. 1) measured along the abscissa, from the point $x = 0$ to the nearest zero value of the sine function.

4. Properties of the Sine Function. — There exist two properties of the sine function which deserve special mention. These are (a) periodicity and (b) bisymmetry.

(a) *Periodicity.* — In so far as all the circular functions are concerned:

$$\alpha = \alpha + 2n\pi \quad (3)$$

Hence

$$e = E_m \sin(x + \alpha) = E_m \sin(x + \alpha + 2n\pi) \quad (4)$$

This means that any two instantaneous values (for example, e_2 Fig. 1) that are 2π radians apart are equal. In other words the function has a period of 2π or is a periodic function.

Due to its periodicity a sine function assumes various values in an interval of 2π radians but always returns to the starting value at the end of that interval. For example the couples of points a and b , c and d ,

f and g (Fig. 1) are respectively 2π radians apart. Hence the values of the sine function at each couple of these points are identical.

The set of values which a periodic function assumes, in any one period, constitutes a *cycle*. Thus the set of negative and positive values (Fig. 1) which the sine function assumes in the interval $0 \leq (x + \alpha) \leq 2\pi$ constitute one cycle.

The term cycle is so often used in engineering that a symbol has been adopted for it. It is a small sine wave ($\sim$).

(b) *Bisymmetry*. — As a consequence of the relation $\sin \beta = \sin (\pi - \beta)$, there exist, within any one period, two arguments (β , and $\pi - \beta$) each of which gives the same instantaneous value of the sine function. For lack of a better name this property shall be termed *bisymmetry*.

Referring to Fig. 1, the instantaneous values e_1 and e_2 are equal. The difference between the two values is that the slope of the curve at the first point is positive and at the second negative. Hence giving the instantaneous value of a sine wave is not sufficient to determine a point on the curve. Some additional information (for example, the slope of the wave) must be supplied.

5. Average and Effective Values. — Frequent reference is made in the text to the average and effective values of a sine function. We shall therefore define these two terms.

(a) *Average Value E_a* . — In general the average value of any function between two limits θ and β is the value of the mean ordinate between these limits. The mathematical expression for this statement is:

$$E_a = \left(\frac{1}{\theta - \beta} \right) \int_{\beta}^{\theta} e \, dx \quad (5)$$

Applying Eq. (5) to a sine function we obtain

$$\begin{aligned} E_a &= \left(\frac{1}{\theta - \beta} \right) \int_{\beta}^{\theta} E_m \sin (x + \alpha) \, dx \\ &= \left(\frac{E_m}{\theta - \beta} \right) [\cos (\beta + \alpha) - \cos (\theta + \alpha)] \end{aligned} \quad (6)$$

It is evident from an inspection of Eq. (6) and Fig. 3 that the average value of a sine wave varies depending upon the limits θ and β . For many purposes, however, the sine wave is integrated from its zero value to an immediately succeeding π value. Thus in Fig. 3 the limits are taken from $x = -\alpha$ to $x = (\pi - \alpha)$. Substituting these limits in Eq. (6) and simplifying we have:

$$E_a = (E_m/\pi) [\cos 0 - \cos \pi] = 2 E_m/\pi \quad (7)$$

(b) *Effective Value E.* — The effective value E of a periodic function between the limits β and θ is the square root of the area under the squared curve divided by the base.

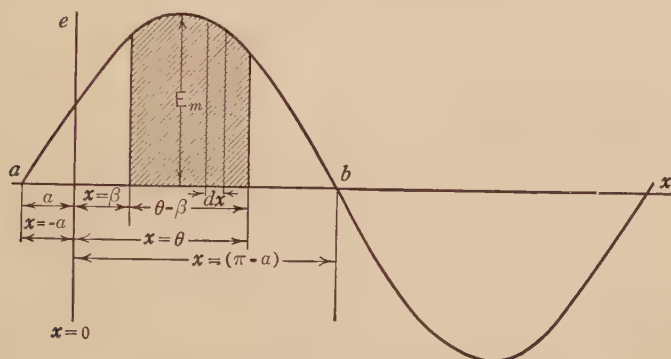


FIG. 3. Average value of a sine function.

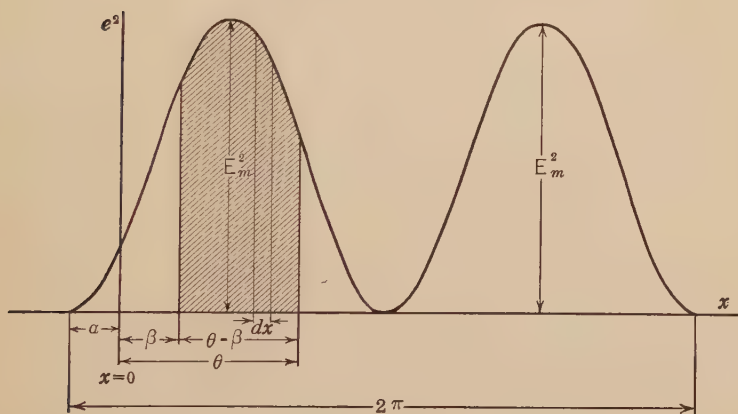


FIG. 4. Effective value of a sine function.

Thus let the curve, Fig. 4, be obtained by squaring the ordinates of the curve Fig. 1. The effective value of the sine wave between the limits θ and β is the square root of the quotient obtained by dividing the value of the shaded area by the base $\theta - \beta$. The mathematical expression for this operation is

$$E = \left[\int_{\beta}^{\theta} e^2 dx / (\theta - \beta) \right]^{\frac{1}{2}} \quad (8)$$

Applying Eq. (8) to a sine function having the form of Eq. (1) we have:

$$E = \left[\int_{\beta}^{\theta} E_m^2 \sin^2 (x + \alpha) dx / (\theta - \beta) \right]^{\frac{1}{2}} \quad (9)$$

Here again the effective value of a sine function will vary depending upon the limits β and θ . For most purposes however the limits are taken as 0 and 2π . Substituting these limits in Eq. (9) we have:

$$E = E_m \left[(1/2\pi) \int_0^{2\pi} \sin^2(x + \alpha) dx \right]^{\frac{1}{2}} = E_m / \sqrt{2} \quad (10)$$

(c) *Amplitude and Form Factors.* — The two values defined above have given rise to two factors known as the amplitude and form factors:

The *Amplitude factor* (k_a) is the ratio of the maximum to the effective value of a periodic wave or:

$$k_a = E_m / E \quad (11)$$

For sine wave

$$k_a = \sqrt{2}$$

The *Form Factor* (k_f) is the ratio of the effective value to the average value of a periodic wave or:

$$k_f = E / E_a \quad (12)$$

For sine waves

$$k_f = \pi / (2\sqrt{2}) \cong 1.11.$$

6. Lead, Lag, and Phase. — Lead means to be ahead and lag means to be behind. Graphically (see Fig. 5) a sine wave leads the origin if its zero value (which is nearest to the origin) occurs before the point $x = 0$ (see curve *a* Fig. 5). A curve lags the origin if its zero value (which is

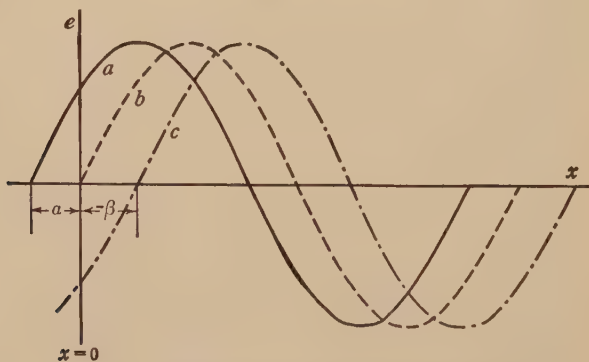


FIG. 5. The concepts of lead, lag, and phase.

nearest to the origin) occurs after the point $x = 0$ (see curve *c* Fig. 5). Finally a sine function is in phase with the origin if its zero value coincides with $x = 0$. (See curve *b* Fig. 5.)

The angle of lead or lag (or the *phase angle* or the *Initial Angle*) is measured from the point $x = 0$ to the nearest zero value of the sine wave.

The terms lead and lag are relative. Thus if curve *a*, Fig. 5, leads the origin then the origin lags curve *a*. These two statements have the same meaning and are interchangeable. Again if curve *a* leads the origin by an angle α and curve *c* lags it by $-\beta$, then curve *a* leads *c* by an angle $(\alpha + \beta)$.

When it is desired to give the phase angle without any reference to lead or lag the term *Out of Phase* is used. Thus in Fig. 5 the curves *a* and *b* are α degrees out of phase and the curves *a* and *c* are $(\alpha + \beta)$ degrees out of phase.

It is very convenient to be able to state whether a sine function leads, lags, or is in phase with the origin by a mere inspection of the function and without actually plotting it. The value of the initial angle α (Eq. 1) determines whether the nearest zero value of a sine function occurs before or after the point $x = 0$. In general, a sine function leads the origin (curve *a*, Fig. 5) if the initial angle is such that $0 < \alpha < \pi$ and lags (curve *c* Fig. 5) if the initial angle is such that $\pi < \alpha < 2\pi$. If $\alpha = 0$ or $\alpha = 2n\pi$ then the function is in phase with the origin. The use of either term lead or lag is justifiable when $\alpha = (2n + 1)\pi$.

7. Addition of Sine Functions. — Let it be required to add the two sine functions represented by curves *a* and *b*, Fig. 6. This operation can be done either graphically or analytically. The graphic method

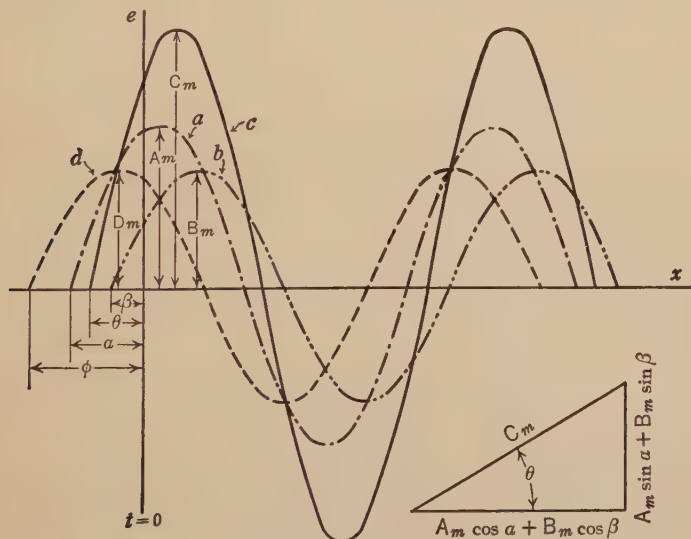


Fig. 6

Fig. 7

FIG. 6. Addition and subtraction of sine functions.

FIG. 7. The amplitude and phase angle of the sine wave *c* which is the sum of the two sine waves *a* and *b*.

consists in adding the ordinates point by point and then plotting their sum, which is curve c (a sine curve). Analytically the sum may be found as follows.

$$\text{Let} \quad \left. \begin{aligned} a &= A_m \sin(x + \alpha) & (a) \\ b &= B_m \sin(x + \beta) & (b) \end{aligned} \right\} \quad (13)$$

$$\text{Then} \quad c = a + b = A_m \sin(x + \alpha) + B_m \sin(x + \beta) \quad (14)$$

Expanding $\sin(x + \alpha)$ and $\sin(x + \beta)$ we have

$$\left. \begin{aligned} c &= A_m(\sin x \cos \alpha + \cos x \sin \alpha) + B_m(\sin x \cos \beta + \cos x \sin \beta) & (a) \\ &= \sin x (A_m \cos \alpha + B_m \cos \beta) + \cos x (A_m \sin \alpha + B_m \sin \beta) & (b) \end{aligned} \right\} \quad (15)$$

Now let a right triangle be constructed as shown in Fig. 7. We have:

$$C_m = [(A_m \cos \alpha + B_m \cos \beta)^2 + (A_m \sin \alpha + B_m \sin \beta)^2]^{\frac{1}{2}} \quad (16)$$

$$\tan \theta = (A_m \sin \alpha + B_m \sin \beta) / (A_m \cos \alpha + B_m \cos \beta) \quad (17)$$

Also

$$\left. \begin{aligned} A_m \cos \alpha + B_m \cos \beta &= C_m \cos \theta & (a) \\ A_m \sin \alpha + B_m \sin \beta &= C_m \sin \theta & (b) \end{aligned} \right\} \quad (18)$$

Substituting Eqs. (18a) and (18b) in (15) we have:

$$c = C_m(\sin x \cos \theta + \cos x \sin \theta) = C_m \sin(x + \theta) \quad (19)$$

Eq. (19) shows that the sum of two sine functions (that are functions of the same argument x) is a sine function whose amplitude and initial angle are expressed by Eqs. (16) and (17) respectively.

8. Subtraction of Sine Functions. — The subtraction of sine functions can be accomplished either graphically or analytically. Thus, referring to Fig. 6 the result of graphically subtracting corresponding ordinates of curves a and b is curve d (a sine wave). Analytically the difference between the two sine functions is obtained exactly as was done in Art. 7. In fact, the result can be easily had by changing the sign of the second term in Eq. (14). We thus have:

$$d = a - b = D_m \sin(x + \phi) \quad (20)$$

$$\text{where} \quad D_m = [(A_m \cos \alpha - B_m \cos \beta)^2 + (A_m \sin \alpha - B_m \sin \beta)^2]^{\frac{1}{2}} \quad (21)$$

$$\tan \phi = (A_m \sin \alpha - B_m \sin \beta) / (A_m \cos \alpha - B_m \cos \beta) \quad (22)$$

Thus the difference of two sine waves is also a sine wave having an amplitude expressed by Eq. (21) and an initial angle given by Eq. (22).

9. Multiplication of Sine Functions. — Consider the two sine waves given in Fig. 8a and let it be required to evaluate their product:

$$p = E_m \sin(x + \alpha) I_m \sin(x + \beta) \quad (23)$$

Making use of the trigonometric transformation $\sin a \sin b = (\frac{1}{2}) [\cos (a - b) - \cos (a + b)]$, we have:

$$\begin{aligned} p &= (E_m I_m / 2) [\cos (\alpha - \beta) - \cos (2x + \alpha + \beta)] & (a) \\ &= (E_m I_m / 2) [\cos (\alpha - \beta) - \sin (2x + \alpha + \beta + 90^\circ)] & (b) \end{aligned} \quad (24)$$

Eq. (24) shows that the product of two sine functions gives a constant term, $(E_m I_m / 2) \cos (\alpha - \beta)$, and a sine term, $-(E_m I_m / 2) \sin (2x + \alpha + \beta + 90^\circ)$. This sine term is a function of $2x$ (not of x). The product

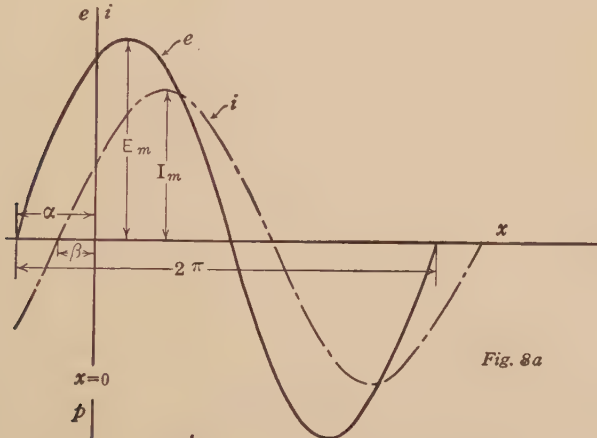


Fig. 8a

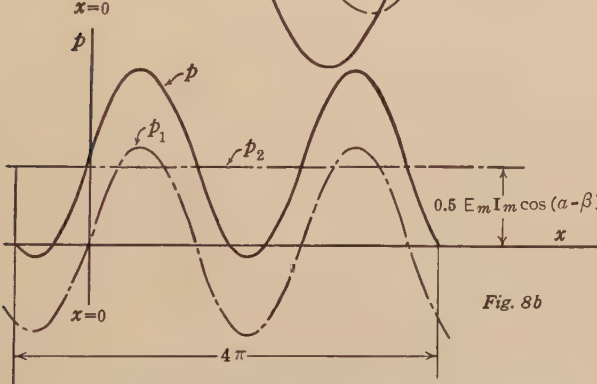


Fig. 8b

FIG. 8a. Two sine functions e and i .FIG. 8b. The product $p = ei$.

is given graphically by curve p , Fig. 8b; and its component parts, namely, the sine function and the constant term, are curves p_1 and p_2 respectively. Observe that curve p completes two periods for every period completed by the sine curves e and i . For this reason it is called a double frequency sine function.

10. Division of Sine Functions. — The division of sine functions is rarely, if ever, encountered in electrical engineering analysis. This is

fortunate because the quotient $q = [E_m \sin(x + \alpha)/I_m \sin(x + \beta)]$ does not lend itself to further simplification.

11. Generation of a Sine Function by a Revolving Vector. — Fig. 9a shows a vector of length E_m and initial argument α . Let this vector revolve at a uniform angular velocity ω such that in time t it moves through an angle x . Then by definition:

$$\omega = x/t \quad \text{or} \quad x = \omega t \quad (25)$$

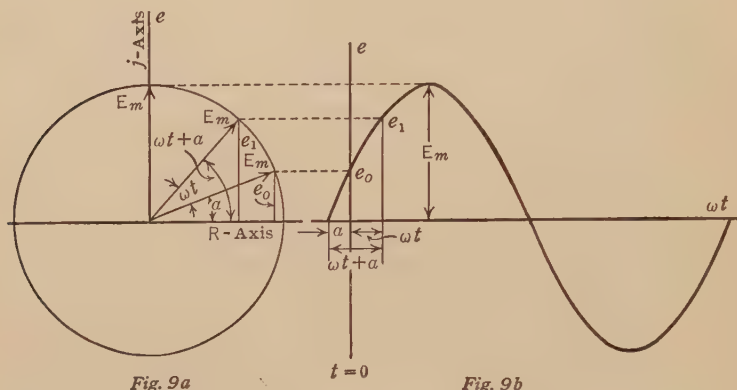


FIG. 9. Generation of the sine function by a revolving vector.

If the period (time required to make one complete revolution or 2π radians) is denoted by T , then:

$$\omega = 2\pi/T \quad (26)$$

consequently

$$\left. \begin{aligned} x &= \omega t = 2\pi t/T & (a) \\ x &= 2\pi ft & (b) \end{aligned} \right\} * \quad (27)$$

or

$$\text{where} \quad f = 1/T \quad (28)$$

f is known as the frequency of the revolving vector and is the number of revolutions or cycles performed per second.

Now consider the projections of the revolving vector on the vertical axis. From the geometry of Fig. 9a these are:

$$e = E_m \sin(x + \alpha) = E_m \sin(\omega t + \alpha) = E_m \sin(2\pi ft + \alpha) \quad (29)$$

But Eq. (29) is simply a sine function which may be represented by the sine wave (Fig. 9b). We therefore conclude that a revolving vector, of modulus E_m and initial argument α , moving at a uniform angular

* Note x may be expressed in radians or degrees depending upon whether π is taken as 3.1416 radians or 180° .

velocity ω has projections along the vertical axis which correspond exactly with the instantaneous values e of a sine function of amplitude E_m argument ωt and initial angle α .

12. The Use of Stationary Vectors in the Theory of Alternating Electric Circuits. — Let two sine waves (Fig. 10) be generated by two revolving vectors having the same angular velocity ω . Let the initial angles of the first and second be α and $-\beta$ respectively. It is evident that as long as the vectors move at the same velocity the angle between them will always be $(\alpha + \beta)$. Now in many applications of sine func-

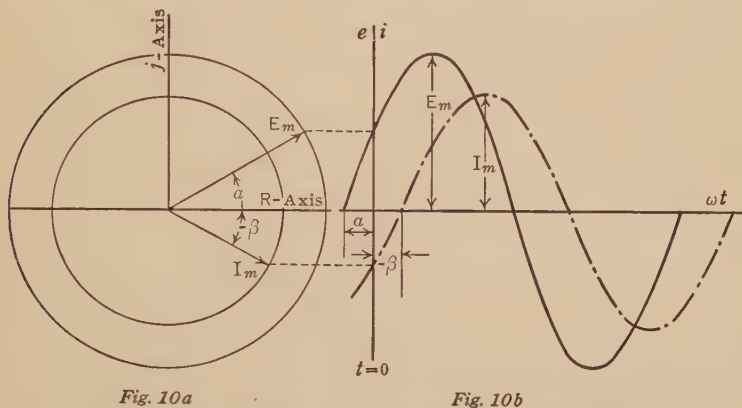


Fig. 10a

Fig. 10b

FIG. 10. Two sine waves and the stationary vectors representing them.

tions to alternating electric circuits this phase angle is sought. As this angle is the same no matter whether the vectors revolve or are stationary and as stationary vectors lend themselves easily to analytical treatments through complex numbers the revolving vector is replaced by a stationary vector. Again, under steady conditions, many of the phenomena that occur in alternating electric circuits are such that what is true of any one instant is also true of all other instants. This furnishes a further justification for replacing revolving vectors by stationary ones which may be drawn for any arbitrary instant.

In representing sine functions by stationary vectors the following facts should be remembered:

- (1) Stationary vectors may be used if and only if all the sine functions entering into the problem are of the same frequency f .
- (2) The phenomena must be such that what is true of one instant is also true of all instants (steady state conditions must prevail).
- (3) The vector is so plotted that it makes an angle, with the reference axis, equal to the initial angle of the sine function. This

angle is the argument of the stationary vector. It is, hence, the direction-angle between the stationary vector and the reference axis. Thus, a vector leads the reference axis if it has an argument α such that $0 < \alpha < \pi$. On the other hand a vector lags the reference axis if its argument is such that $-\pi < \alpha < 0$. A vector is in phase with the R -axis if $\alpha = 0$.

(4) The angle that the vector makes with the reference axis is measured counterclockwise from the R -axis to the vector. If the angle is measured clockwise from the R -axis to the vector it is considered a negative angle. Thus the angle β , Fig. 10a, is a negative angle.

(5) The length of the vector is made proportional to the effective value E , instead of the amplitude E_m , of the sine function. This departure from the strictly correct representation described above may be, at first, confusing. Its justification, however, for practical purposes, becomes apparent when it is noted that meters read effective values rather than amplitudes and most of the computations for electric circuits are based on effective values. Again, for sine waves, the relation between the effective value and the amplitude is definite (see Eq. 10). Hence a change of scale makes the same vector represent either E or E_m .

(6) The vectors representing sine functions possess all the properties of vectors outlined in Chapter II. As all vectors used in this text are coplanar, complex numbers can be used to define their moduli and arguments.

In order to make these facts clear we shall give the instantaneous expressions for the sine curves drawn in Fig. 10b, and then give the complex expressions for the stationary vectors representing them. These are:

$$\left. \begin{aligned} e &= E_m \sin(\omega t + \alpha) & (a) \\ i &= I_m \sin(\omega t - \beta) & (b) \end{aligned} \right\} \quad (30)$$

$$\left. \begin{aligned} \mathbf{E} &= E e^{j\alpha} = (E_m/\sqrt{2}) e^{j\alpha} & (a) \\ \mathbf{I} &= I e^{-j\beta} = (I_m/\sqrt{2}) e^{-j\beta} & (b) \end{aligned} \right\} \quad (31)$$

It should be here noted that Eqs. (30) and (31) are reversible. In other words, given Eqs. (30) the student can easily write Eqs. (31) and vice versa.

13. Some Advantages of the Vector Representation of Sine Functions.

— The representation of sine functions by vectors possesses advantages which simplify the algebraic operations and render the physical phenomena clearer to the student. The demonstration of the latter aspect is

reserved to future chapters. We shall here show how the addition and subtraction of sine functions is simplified through the use of vectors.

Let it be required to add the three sine waves:

$$\left. \begin{aligned} a &= A_m \sin (\omega t + \alpha) & (a) \\ b &= B_m \sin (\omega t + \beta) & (b) \\ c &= C_m \sin (\omega t + \gamma) & (c) \end{aligned} \right\} \quad (32)$$

Vectorially these sine waves become:

$$\left. \begin{aligned} \mathbf{A} &= A \epsilon^{j\alpha} = a' + ja'' & (a) \\ \mathbf{B} &= B \epsilon^{j\beta} = b' + jb'' & (b) \\ \mathbf{C} &= C \epsilon^{j\gamma} = c' + jc'' & (c) \end{aligned} \right\} \quad (33)$$

Whence:

$$\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C} = (a' + b' + c') + j(a'' + b'' + c'') = d' + jd'' = D \epsilon^{j\theta}$$

Then the sum of the three sine waves is a sine wave having an amplitude $D_m = D \sqrt{2}$ and an initial angle θ . The instantaneous value of this wave is:

$$d = D_m \sin (\omega t + \theta) \quad (34)$$

Caution. — As the product of two sine functions yields a constant term and a double frequency sine term (see Art. 9) and as the frequency is entirely omitted in the representation of a sine function by a stationary vector, the student is hereby warned to avoid the use of vectors in multiplying two sine functions.

14. Cosine Functions. — Cosine functions are met with rather frequently in this text. A separate treatment of these functions becomes superfluous when it is remembered that:

$$\cos (x + \alpha) = \sin [x + \alpha + (\pi/2)] \quad (35)$$

Thus a cosine function can be always converted to a sine function by adding $(\pi/2)$ to the initial angle of the cosine function.

PROBLEMS

1. A sine function has the following constants: $E_m = 150$ and $\alpha = \pi/3$. Write its equation.

2. Plot the sine wave of the function, $e = 150 \sin (x + 25^\circ)$ in rectangular coordinates and in polar coordinates. Indicate on the graphs the initial value e_0 and the initial angle $\alpha = 25^\circ$.

3a. Show on the curves obtained in problem 2 the following values of the sine function:

(1) Amplitude, (2) zero values, (3) π values, and (4) initial value.

3b. A sine function has the form $e = 270 \sin(x - \pi/4)$. Give its initial value and state whether the slope of the curve at that point is positive or negative. Sketch the sine function.

3c. The initial value of a sine function is 25 and its amplitude is 150. Moreover the slope of the curve at e_0 is positive. Write the equation of the function and give its graph.

3d. The equation of a sine function is $e = E_m \sin(x + 30^\circ)$. If the initial value of the function is 75 what is the amplitude?

4a. Two sine functions have the same amplitudes and the same initial values which are not equal to zero. The slope of the curve at the initial value is positive for the first and negative for the second.

(1) Give the equations of the two functions.

(2) Sketch the curves for the two functions on the same paper showing the point $x = 0$.

4b. Draw the sine function given in problem 3b for two periods and indicate at least ten points on the curve any two of which are one period apart. Now write the instantaneous value expressions for each of these points.

4c. The equation of a function is $e = E_m \sin(x + 60^\circ)$. Give the expression for a point on the curve which has the same value as the initial value. Sketch the curve and indicate the two points.

5a. Certain electric circuits are acted upon by only a portion of the sine voltage. Let this portion be included between the values $\omega t = \alpha$ and $\omega t = (\pi - \alpha)$.

Let the expression for the instantaneous value of the voltage be:

$$e = E_m \sin \omega t$$

(1) Evaluate the average and effective values of the acting voltage.

(2) Compute the average and effective values if $\alpha = 70^\circ$.

(3) Give the amplitude and form factors for the acting voltage.

5b. Assume that, for some purpose, a continuous and an alternating Emf. source are connected in series such that the instantaneous value of the combination is given by the expression

$$e = E_1 + E_m \sin(\omega t + \alpha)$$

Find the average and effective values of this combination for the limits $\omega t = -\alpha$ and $\omega t = (\pi - \alpha)$.

5c. Give at least six different limits such that when two of them are used, the average value of the sine function, $e = E_m \sin(x + 30^\circ)$, over the range bounded by these limits, is zero.

5d. Show, by actually substituting different limits in Eq. (9), that the effective value of a sine function depends upon the limits chosen.

6a. Write the instantaneous value expression for a sine wave which:

(1) Leads the origin by 60° .

(2) Lags the origin by 60° .

(3) Leads the origin by 120° .

(4) Lags the origin by 120° .

Sketch these waves showing the point $x = 0$.

6b. Give the phase angles between the following curves in problem 6a and indicate which is the leading curve:

- (a) Curve 1 and 2.
- (b) Curve 1 and 3.
- (c) Curve 2 and 3.
- (d) Curve 3 and 4.
- (e) Curve 2 and 4.

7a. The equations of two sine functions are:

$$e_1 = 150 \sin (x + 25^\circ)$$

$$e_2 = 27 \pi \sin (x + 75^\circ)$$

Give the expression for the sine function $e = e_1 + e_2$.

7b. Show that the sum of the instantaneous values of three sine waves (of equal amplitudes), that are equally displaced from each other by a phase angle of 120° is zero.

7c. Show that the sum of the instantaneous values of four sine functions, of equal amplitudes, that are equally displaced from each other by a phase angle of 90° , is zero.

8. Evaluate $e' = e_1 - e_2$ and $e'' = e_2 - e_1$ where e_1 and e_2 are given in problem 7a.

9. Evaluate the product $p = e_1 e_2$ where e_1 and e_2 are expressed in problem 7a.

10. Plot the curve for the quotient (e_1/e_2) , where e_1 and e_2 are given in problem 7a.

11a. What is the angular velocity ω of a voltage wave having a frequency of 60 cycles per second?

11b. The equation for the instantaneous value of a sine current is

$$i = 15 \sin (2 \pi 60 t - 25^\circ)$$

(a) Draw the sine wave showing the point $t = 0$.

(b) What is the value of the current at the instant $t = 0$?

11c. A sine voltage has an amplitude of 150 volts and a value of 40 volts at an arbitrarily chosen instant of time $t = 0$. Moreover the slope of the wave at this initial point is negative.

(a) Write the analytical expression for the voltage wave.

(b) Sketch the sine wave indicating the instant $t = 0$.

11d. Indicate the positions of the revolving vector, which represents the current in problem 11b, for the following instants of time:

(a) $t = 0$, (b) $t = T/4$, (c) $t = 3 T/8$.

(d) $t = T/2$, (e) $t = 3 T/4$, (f) $t = T$.

12a. Draw the stationary vector which represents the voltage specified in problem 11c and indicate the angle it makes with the reference axis.

12b. Write the complex expressions for the current and voltage given in problems 11b and 11c respectively.

12c. The complex expressions for the vectors representing two voltage waves are:

$$\mathbf{E}_{12} = 70 \angle 30^\circ$$

$$\mathbf{E}_{23} = 40 \text{ cis } 45^\circ$$

(a) Which of the two voltages leads?

(b) What is the angle of lead between the two voltages?

- (c) By how many degrees does E_{12} lead the reference axis?
- (d) Draw a vector diagram of the two voltages.
- (e) Give the sine functions represented by the two vectors.

13a. Give the instantaneous value of the sum of the two voltages given in problem 12c.

13b. In a polyphase star connected circuit, the instantaneous value of the line voltage e_{12} is:

$$e_{12} = e_{02} - e_{01}$$

Let:

$$e_{01} = 150 \sin (\omega t + 30^\circ)$$

$$e_{02} = 150 \sin (\omega t - 90^\circ)$$

- (a) Draw to scale the vectors E_{01} and E_{02} and show their positions with respect to the reference axis.
- (b) Give the complex expressions, in the orthogonal form, for E_{01} and E_{02} .
- (c) Find $E_{12} = E_{02} - E_{01}$.
- (d) Change the complex expression obtained for E_{12} into the exponential form.
- (e) Write the instantaneous expression for e_{12} .

13c. Three currents have the following instantaneous values:

$$i_1 = 15 \sin (\omega t + 25^\circ)$$

$$i_2 = 20 \sin (\omega t - 30^\circ)$$

$$i_3 = 30 \sin (\omega t + 40^\circ)$$

- (1) Give the complex expressions for the vectors representing these currents.
- (2) Find $I = I_1 + I_2 + I_3$.
- (3) Express I in the circular form.
- (4) Write the sine function which is represented by the vector I . This is the sum of the three sine currents.

13d. Give the proofs required in problems 7b and 7c by representing the sine functions by stationary vectors.

14a. A cosine function has the form: $e = E_m \cos (x + 30^\circ)$.

- (1) Referring to the Table I, Appendix II, plot the function in rectangular coordinates.
- (2) Convert the cosine to a sine function.
- (3) Plot the sine function so obtained and convince yourself that the two curves obtained in parts (1) and (3) are identical.

14b. Two functions are:

$$e_1 = E_m \cos (\omega t + \alpha)$$

$$e_2 = E_m \sin (\omega t + \alpha)$$

- (1) What is the phase angle between these two functions?
- (2) Which of the two functions leads?
- (3) Sketch the graphs of e_1 and e_2 .

CHAPTER IV

PHYSICAL QUANTITIES, UNITS AND DIMENSIONS

1. Introduction. — The course of scientific progress may be divided into the following stages:

1. Discovery of a phenomenon.
2. Determination of the numerical relations existing between the factors which enter into the phenomenon.
3. Correlating the phenomenon with others already known.
4. Application of the discovery either to promote the progress of science or the welfare of mankind.

Consider, as an illustration, Faraday's Law of Electromagnetic Induction. The first stage in the research of this noted physicist was to discover the phenomenon that the motion of a conductor in a magnetic field induces an Emf. in the conductor. The second stage consisted in the establishment of the numerical law that $e = \mathfrak{B}vD$ (see Eq. 1, Chapter V). The third stage was to correlate this phenomenon with the apparently different phenomenon that even with a stationary conductor an Emf. is induced if the magnitude of the field is varied with time (transformer action). The fourth and last stage is the multitudinous applications of this discovery to electrical machinery and electrical appliances.

The engineer is mostly concerned with the last stage. His task is to apply the discoveries of physical laws to promote the welfare of mankind. But such a task can not be accomplished without a thorough understanding of the work of the physicist who is principally interested in the progress of science per se and whose endeavors are directed toward the attainment of the first three stages and part of the last stage in scientific progress.

Now the factors which make up a physical phenomenon are known as physical quantities. Any physical quantity becomes completely defined:

- (a) When its nature is known.
- (b) When its magnitude as compared with a certain standard quantity called a unit is specified.

Thus the physical quantities Emf., resistance, and current are three physical entities having for their units the volt, ohm and ampere re-

spectively. Moreover the magnitude of an Emf., E , may be expressed in terms of the standard volt as E volts.

In the early development of the science of physics the importance of establishing a correlated scientific system of units was not realized. Hence a new unit was assigned to every physical quantity irrespective of whether such a physical quantity was related to others already established or not. Thus it happens that we have the erg and the joule; the ohm, the statohm, and the abohm; the ampere, the statampere and the abampere, etc. This duplication (or rather multiplication) of units has been further aggravated through chauvinism and nationalism. Illustrations of this may be found in the existence of the Weber and Maxwell as units of magnetic flux, the Gilbert and the Ampere-turn as units of magnetomotive force, the Gauss and the Maxwell (per square centimeter) as two names of the identical unit of flux density. Such is the legacy of confusion and duplication that has been handed down to us. The author feels that if the illustrious physicists, whose names we have attempted to perpetuate through the medium of physical units, had any opinion to express relative to the matter; they would have declined the honor in order to avoid confusion in a science which they loved and to which they devoted their lives.

It is needless to assert that physical laws hold true independently of the choice of units provided that a consistent set of units is adopted. At least three consistent sets of physical units are known. These are:

- (a) The Electromagnetic System (abbreviated e.m.s.)
- (b) The Electrostatic System (abbreviated e.s.s.)
- (c) The Practical System (abbreviated p.s.).

The practical system is almost exclusively used by engineers. The physicist, however, makes use of all the three systems. A correlation of the relative magnitudes of the units in the three systems is of extreme importance to the engineer whose work borders on the realm of the physicist. The ratio of the sizes of units in the three systems is given in the last three columns of Table I. The object of this table is four-fold:

- (a) It gives the legend of notations used in the text.
- (b) It establishes the relations between the magnitudes of corresponding units in the three systems.
- (c) It gives the names of the units in the practical system.
- (d) It gives the physical dimensions of the units.

Parts (a) and (b) need no further comments. Regarding part (c) it has been the author's experience to have some students express energy in watts instead of joules or inductance in ohms instead of henrys. The

practical system is used throughout this text. If the student is at all in doubt as to what units the answer should be in, he is kindly requested to refer to the column headed "Name of unit in the practical system."

We shall now proceed to explain the last object of Table I.

2. Physical Dimensions. — All physical units may be divided into two classes:

(a) *Fundamental units.*

(b) *Derived units.*

(a) *Fundamental Units.* — By fundamental units is meant those units which are arbitrarily defined and are not interrelated. Moreover, through these fundamental units, all other physical units may be explicitly defined. Length $[L]$, * mass $[M]$, and time $[T]$ are generally recognized as the fundamental units. The unit of length is the centimeter, that of mass is the gram, and that of time is the second.

In order to define electrical quantities in terms of these units at least one of two auxiliary fundamental units must be added. The two possible units are the magnetic permeability $[\mu]$, and the dielectric permittivity $[\kappa]$.

If the magnetic permeability $[\mu]$ is adopted as an auxiliary fundamental unit then all electric and magnetic units may be explicitly defined in terms of $[L]$, $[M]$, $[T]$ and $[\mu]$. We thus obtain what is known as the *electromagnetic system of units*.

If the dielectric permittivity $[\kappa]$ is chosen as an auxiliary fundamental unit then again all electric and magnetic units may be explicitly defined in terms of $[L]$, $[M]$, $[T]$ and $[\kappa]$. We thus obtain what is known as the *electrostatic system of units*.

Although in Table I both the electromagnetic and electrostatic systems are included yet, for the purposes of this treatment we shall regard $[\mu]$ and $[\kappa]$ as derived units and shall use the electric current $[I]$ and the electric resistance $[R]$ as two auxiliary fundamental units. We shall adopt the ampere as the unit of current and the ohm as the unit of resistance. All electric and magnetic quantities may then be defined in terms of $[L]$, $[T]$, $[I]$ and $[R]$. A system of units is thus obtained which shall be known as the *practical system of units*. Mass does not enter explicitly as a fundamental unit in this system of electric units.

(b) *Derived Units.* — Any physical unit other than the fundamental units and their auxiliaries falls under the class of derived units. Such derived units may be explicitly defined in terms of the fundamental units. We shall give three examples of derived units:

*The use of brackets indicates that the notation is a dimensional notation.

TABLE I.—PHYSICAL UNITS USED IN THIS TEXT AND THEIR DIMENSIONS

Symbols	Physical Quantity	Equation	DIMENSION			Name of Unit in the Practical System	Ratio of Sizes of Units in the Three Systems		
			Electro-magnetic system (e.m.s.)	Electro-static system (e.s.s.)	Practical system (p.s.)		p.s. e.m.s.	p.s. e.s.s.	e.m.s. e.s.s.
L, l	Length	Fundamental Unit	[L]	[L]	[L]	centimeter (cm.)	1	1	1
M	Mass	Fundamental Unit	[M]	[M]	[M]	gram (gr.)	1	1	1
T, t	Time	Fundamental Unit	[T]	[T]	[T]	second (sec.)	1	1	1
μ	Magnetic Permeability	e.m.s.: μ = Fund. Unit e.s.s.: $\mu = 4\pi/\mathcal{K}$ p.s.: $\mu = 4\pi/\mathcal{K}$	[μ]	[$L^{-2}T^2\kappa^{-1}$]	[RTL^{-1}]	henry per cm. cube	10^9	$10^{9\frac{1}{2}}$	ν^{-2}
κ	Dielectric Permittivity	e.s.s.: κ = Fund. Unit e.m.s.: $\kappa = D/\epsilon$ p.s.: $\kappa = D/\epsilon$	[ϵ]	[ϵ]	[$R^{-1}TL^{-1}$]	farad per cm. cube	10^{-9}	$10^{-9\frac{1}{2}}$	ν^2
I, i	Electric Current	p.s.: i = Fund. Unit e.m.s.: $\mathcal{H} = (i dl/\epsilon^2)$ e.s.s.: $i = (q/t)$	[$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$]	[$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\kappa^{-\frac{1}{2}}$]	[I]	ampere (Amp.)	10^{-1}	$10^{-1\nu}$	ν
R, r	Electric Resistance	p.s.: r = Fund. Unit e.m.s.: $r = (e/i)$ e.s.s.: $r = (e/i)$	[$LT^{-1}\mu$]	[$L^{-1}T\kappa^{-1}$]	[R]	ohm	10^9	$10^{9\frac{1}{2}}$	ν^{-2}
A	Area	$A = L_1L_2$	[L ²]	[L ²]	[L ²]	square cm. (sq. cm.)	1	1	1
V	Volume	$V = L_1L_2L_3$	[L ³]	[L ³]	[L ³]	cubic cm. (cu. cm.)	1	1	1
α, β	Plane Angles	$\tan \alpha = (L_1/L_2)$	[θ]	[θ]	[θ]	degree (deg.) radian (rad.)	1	1	1
v	Linear Velocity	$v = (dl/dt)$	[LT^{-1}]	[LT^{-1}]	[LT^{-1}]	cm. per sec.	1	1	1
ω	Angular Velocity	$\omega = 2\pi f$	[T^{-1}]	[T^{-1}]	[T^{-1}]	deg. per sec. rad. per sec.	1	1	1
U	Current Density	$U = (I/A)$	[$L^{-3}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$]	[$L^{-3}M^{\frac{1}{2}}T^{-1}\kappa^{-\frac{1}{2}}$]	[L^{-3}]	amp. per sq. cm.	10^{-1}	$10^{-1\nu}$	ν
Q, q	Quantity of Electricity or Electric Charge	$q = it$	[$L^{\frac{1}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$]	[$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\kappa^{-\frac{1}{2}}$]	[IT]	coulomb	10^{-1}	$10^{-1\nu}$	ν
$\mathcal{E}, e$	Electromotive Force (Emf.)	e.m.s.: $w = eit$ e.s.s.: $w = eit$ p.s.: $w = eit$	[$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$]	[$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\kappa^{\frac{1}{2}}$]	[RI]	volt (V.)	10^8	$10^{8\nu-1}$	ν^{-1}
D	Surface Density of Charge	$D = (Q/A)$	[$L^{-3}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$]	[$L^{-3}M^{\frac{1}{2}}T^{-1}\kappa^{\frac{1}{2}}$]	[ITL^{-2}]	coulomb per sq. cm.	10^{-1}	$10^{-1\nu}$	ν

ϵ	Potential Gradient, Electric Intensity or Electric Stress	$e = (de/dt)$	$[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-\frac{1}{2}} \mu^{\frac{1}{2}}]$	$[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-\frac{1}{2}} \kappa^{-\frac{1}{2}}]$	$[RIL^{-1}]$	volt per cm.	10^8	10^9	ν^{-1}
C	Capacitance (Permittance)	$C = (q/e)$	$[L^{-1} T^2 \mu^{-1}]$	$[L\kappa]$	$[R^{-1} T]$	farad ($\mathcal{F}$)	10^{-9}	$10^{-9\frac{1}{2}}$	ν^2
S	Elastance	$S = (1/C)$	$[LT^{-2}\mu]$	$[L^{-1}\kappa^{-1}]$	$[RT^{-1}]$	daraf	10^9	$10^{9\frac{1}{2}}$	ν^{-2}
$\mathcal{L}$	Inductance or Coefficient of self-induction	$e = -\mathcal{L}(di/dt)$	$[L\mu]$	$[L^{-1} T^2 \kappa^{-1}]$	$[RT]$	henry	10^9	$10^{9\frac{1}{2}}$	ν^2
$\mathfrak{M}$	Mutual-inductance or Coefficient of Mutual-induction	$e_{12} = \mathfrak{M}(di_1/dt)$	$[L\mu]$	$[L^{-1} T^2 \kappa^{-1}]$	$[RT]$	henry	10^9	$10^{9\frac{1}{2}}$	ν^2
X, x	Reactance	$z = 2\pi f\mathcal{L} = (1/2\pi fC)$	$[LT^{-1}\mu]$	$[L^{-1} T\kappa^{-1}]$	$[R]$	ohm	10^9	$10^{9\frac{1}{2}}$	ν^{-2}
Z, z	Impedance	$z = \sqrt{r^2 + z^2}$	$[LT^{-1}\mu]$	$[L^{-1} T\kappa^{-1}]$	$[R]$	ohm	10^9	$10^{9\frac{1}{2}}$	ν^{-2}
B, b	Susceptance	$b = (1/2\pi f\mathcal{L}) = 2\pi fC$	$[L^{-1} T\mu^{-1}]$	$[LT^{-1}\kappa]$	$[R^{-1}]$	mho	10^{-9}	$10^{-9\frac{1}{2}}$	ν^2
G, g	Conductance	$G = (1/R)$	$[L^{-1} T\mu^{-1}]$	$[LT^{-1}\kappa]$	$[R^{-1}]$	mho	10^{-9}	$10^{-9\frac{1}{2}}$	ν^2
Y, y	Admittance	$y = \sqrt{g^2 + b^2}$	$[L^{-1} T\mu^{-1}]$	$[LT^{-1}\kappa]$	$[R^{-1}]$	mho	10^{-9}	$10^{-9\frac{1}{2}}$	ν^2
ρ	Electric Resistivity	$\rho = (\epsilon/U)$	$[L^2 T^{-1}\mu]$	$[T\kappa^{-1}]$	$[RL]$	ohm per cm. cube	10^9	$10^{9\frac{1}{2}}$	ν^{-2}
γ	Electric Conductivity	$\gamma = (1/\rho)$	$[L^{-2} T\mu^{-1}]$	$[T^{-1}\kappa]$	$[R^{-1} L^{-1}]$	mho per cm. cube	10^{-9}	$10^{-9\frac{1}{2}}$	ν^2
ν	Magnetic Reluctivity	$\nu = (1/\mu)$	$[\mu^{-1}]$	$[L^{-2} T^{-1}\kappa]$	$[R^{-1} T^{-1} L]$	Yrueh per cm. cube	10^{-9}	$10^{-9\frac{1}{2}}$	ν^2
σ	Dielectric Elastivity	$\sigma = (1/(\nu\kappa))$	$[L^2 T^{-1}\mu]$	$[\kappa^{-1}]$	$[RT^{-1} L]$	daraf per cm. cube	10^9	$10^{9\frac{1}{2}}$	ν^{-2}
Φ, ϕ	Magnetic Flux	$e = (d\phi/dt)$	$[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-\frac{1}{2}} \mu^{\frac{1}{2}}]$	$[L^{\frac{1}{2}} M^{\frac{1}{2}} \kappa^{-\frac{1}{2}}]$	$[RIT]$	weber	10^8	$10^{8\frac{1}{2}}$	ν^{-1}
$\mathfrak{B}$	Magnetic Flux Density or Magnetic Induction	$\mathfrak{B} = (\phi/A)$	$[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-\frac{1}{2}} \mu^{\frac{1}{2}}]$	$[L^{-\frac{1}{2}} M^{\frac{1}{2}} \kappa^{-\frac{1}{2}}]$	$[RITL^{-2}]$	weber per sq. cm.	10^8	$10^{8\frac{1}{2}}$	ν^{-1}
$\mathcal{F}$	Magnetomotive Force (Mmf.)	$\mathcal{F} = KI$ $K = \text{constant}$	$[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-\frac{1}{2}} \mu^{\frac{1}{2}}]$	$[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-\frac{1}{2}} \kappa^{\frac{1}{2}}]$	$[I]$	ampere-turn (A.T.)	10^{-1}	$10^{-1\frac{1}{2}}$	ν
$\mathcal{H}$	Magnetic Intensity or Mmf. Gradient	$\mathcal{H} = (d\mathcal{F}/dl)$	$[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-\frac{1}{2}} \mu^{\frac{1}{2}}]$	$[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-\frac{1}{2}} \kappa^{\frac{1}{2}}]$	$[IL^{-1}]$	A.T. per cm.	10^{-1}	$10^{-1\frac{1}{2}}$	ν
f	Frequency	$f = 1/T$	$[T^{-1}]$	$[T^{-1}]$	$[T^{-1}]$	cycle per sec $\omega/\text{sec.}$	1	1	1
P	Electric Power	$P = EI = (W/T)$ $\left. \begin{array}{l} \text{e.m.s.} \\ \text{e.a.s.} \end{array} \right\} W = LM(d\phi/dt)$ $\left. \begin{array}{l} \text{e.m.s.} \\ \text{p.s.} \end{array} \right\} W = eil$	$[L^2 MT^{-3}]$	$[L^2 MT^{-3}]$	$[R]^2$	watt	10^7	10^7	1
W, w	Electric Energy		$[L^2 MT^{-2}]$	$[L^2 MT^{-2}]$	$[R]^2 T$	joule	10^7	10^7	1
$p.f.$	Power Factor	$p.f. = \cos \phi$	$[1]$	$[1]$	$[1]$	number	1	1	1

The names of units in the electromagnetic and electrostatic systems are derived from those in the practical system by prefixing "ab" and "stat" to the name. Thus Abvolt and Statvolt are the units of Emf. in the electromagnetic and electrostatic systems respectively.
 ν in the last three columns of this table is the velocity of light ($\nu = 3 \times 10^{10}$ cm./sec.).

(1) The unit of volume $[V]$ is the volume of a cube having sides of unit length or:

$$[V] = [L^3] \quad (1)$$

Hence the unit of volume is the *centimeter cube*.

(2) By Ohm's law a unit Emf. (the volt) may be defined as that unvarying Emf. which is necessary to force a current of one ampere through a resistance of one ohm or:

$$[E] = [RI] \quad (2)$$

In other words the unit of Emf. is the *ampere-ohm or volt*.

By Faraday's Law of Electromagnetic Induction we have:

$$e = \mathcal{L}(di/dt) \quad (3)$$

whence

$$[\mathcal{L}] = [ETI^{-1}] \quad (4)$$

Substituting Eq. (2) in (4) we have:

$$[\mathcal{L}] = [RT] \quad (5)$$

Thus the unit of inductance is the *ohm-second or henry*.

This mode of defining derived physical units in terms of the fundamental physical units is known as the *dimensional method* and the algebraic formula through which a derived unit is expressed is known as the *dimensional formula* of the unit or the *physical dimension* of the unit.

The physical dimensions of the units used in this text are given in Table I. The student is urged to study this table and check the physical dimensions given therein in the manner outlined above.

3. Some Uses of Dimensional Formulas. — Two principal uses of dimensional formulas may be mentioned:

- (a) Checking equations dimensionally for the purpose of detecting errors either in the reasoning or in the intermediate mathematical steps leading to the result.
- (b) Deriving physical laws, solving physical problems, or predicting physical phenomena.

(a) *Checking Equations Dimensionally.* — The dimensional checking of an equation or a formula consists in finding out whether the physical dimensions of all the terms entering into an equation are identical. The reason is that two units of different physical dimensions can neither be added nor subtracted and hence an equation involving the addition or subtraction of different physical quantities is fundamentally incorrect.

As an example of this use of dimensional formulas consider the following equation expressing the effective value of the secondary current in

terms of the characteristics of two inductively coupled circuits (see Eq. 22, Chapter XI).

$$I_2 = X_m E_1 / [Z_1^2 Z_2^2 + 2 X_m^2 (R_1 R_2 - X_1 X_2) + X_m^4]^{1/2} \quad (6)$$

The validity of Eq. (6) is not at once apparent and it would be desirable to check it dimensionally. Substituting the physical dimensions (Table I) of all the quantities entering into this equation we have:

$$\begin{aligned} [I] &= [R] [RI] / \{ [R^4] + [R^2] [R^2] + [R^4] \}^{1/2} \quad (a) \\ &= [R^2 I] / [R^2] = [I] \quad (b) \end{aligned} \quad (7)$$

which shows that the equation is at least dimensionally correct. Now suppose that through some error the term X_m^4 in the denominator of Eq. (6) were replaced by (say) X_m^2 . Then the dimensional formula (7a) becomes:

$$[I] = [R^2 I] / \{ [R^4] + [R^4] + [R^2] \}^{1/2} \quad (8)$$

This equation is evidently incorrect because in the denominator we have three terms added the first two of which have the same physical dimension $[R^4]$. The last, however, $[R^2]$ differs dimensionally from them. Hence the three terms in the denominator can not be added. We at once conclude that a mistake occurs somewhere. In order to ascertain the source of the error we note that the physical dimension of the right-hand side of Eq. (8) must be I . Such can be the case if and only if the term $[R^2]$ were $[R^4]$. The mistake then is in the power of X_m which should be X_m^4 .

As a corollary to this use of physical dimensions we would state that all exponents must have the physical dimension $[1]$ of a number. If, therefore, the exponent happens to possess any except the dimension of a number then an error must have been committed. Consider, as an illustration, the familiar equation of current in a series circuit of R , $\mathcal{L}$ and E . This is:

$$i = (E/R) (1 - e^{-Rt/\mathcal{L}}) \quad (9)$$

The dimension of $Rt/\mathcal{L}$ must be $[1]$ which it is.

(b) *Deriving Physical Laws, Solving Physical Problems or Predicting Physical Phenomena.* — As an illustration of this let it be given that the energy stored in a plane condenser having plates of a given area A depends only upon:

- (a) The permittivity κ of the dielectric between the plates.
- (b) The thickness of the dielectric, that is, the distance (d) between the plates.
- (c) The applied Emf. E .
- (d) Directly upon the area A of the plates.

Let it be required to find how each of these factors influences the stored energy.

We have by assumption:

$$W = N A \kappa^a d^b E^c \quad (10)$$

$$\begin{array}{l} \text{where (see Table I)} \end{array} \left. \begin{array}{l} [W] = [R I^2 T] \\ [N] = [1] \\ [A] = [L^2] \\ [\kappa] = [R^{-1} T L^{-1}] \\ [d] = [L] \\ [E] = [R I] \end{array} \right\} \quad (11)$$

a, b, c , are powers to be determined.

Substituting the physical dimensions from Eqs. (11) in Eq. (10) we have:

$$[R I^2 T] = [L^2] [R^{-1} T L^{-1}]^a [L^b] [R^c I^c] \quad (12)$$

whence

$$\left. \begin{array}{lll} [R] = [R^{c-a}] & \therefore & c - a = 1 \\ [I^2] = [I^c] & \therefore & c = 2 \\ [T] = [T^a] & \therefore & a = 1 \\ [1] = [L^{2-a+b}] & \therefore & 2 - a + b = 0 \end{array} \right\} \quad (13)$$

Substituting the values of a, b , and c from Eqs. (13) in Eq. (10) we have:

$$W = N A \kappa E^2 / d \quad (14)$$

In other words the energy stored in a plane condenser of a given plate-area varies directly as the area, the permittivity and the square of the applied Emf. and inversely as the thickness of the dielectric.

As another illustration of this use of physical dimensions we shall explain how one of the most striking discoveries of theoretical physics was made.

Referring to the physical dimension of current in the electrostatic and the electromagnetic systems we have:

$$[I] = [L^{1/2} M^{1/2} T^{-1} \mu^{-1/2}] = [L^{3/2} M^{1/2} T^{-2} \kappa^{1/2}] \quad (15)$$

from which

$$\begin{array}{l} [\kappa^{1/2} \mu^{1/2}] = [L^{-1} T] \quad (a) \\ \text{or} \quad 1/\sqrt{\mu \kappa} = [L T^{-1}] \quad (b) \end{array} \quad (16)$$

Now the right-hand side of Eq. (16b) is the physical dimension of velocity, hence $1/\sqrt{\mu \kappa}$ has the physical dimension of velocity. Furthermore the actual value of $1/\sqrt{\mu \kappa}$ is about 3×10^{10} cm./sec. which is approximately the velocity of light. This led Maxwell to conclude that

light is an electromagnetic wave. The results of this discovery have been very far-reaching indeed.

4. **The Practical System of Physical Dimensions vs. The Electromagnetic and Electrostatic Systems.** — Referring to Eq. (15) it will be observed that $[M^{1/2}]$ enters as part of the physical dimension of current in both the e.m.s. and the e.s.s. $[M^{1/2}]$ enters also as a part of the physical dimensions of charge, Emf., and flux. Now $M^{1/2}$ is mathematically the square root of mass. Physically, however, it is meaningless considering our meager knowledge of the concept of mass. Apparently electric and magnetic phenomena are quite distinct and separate from gravitational phenomena. Hence to express one set of phenomena in terms of the other leads to physical concepts that are either absurd or unknown to science. It is for this reason that a deviation from the concept of mass as a fundamental unit in electrical phenomena, and the adoption of the electric current and electric resistance as two arbitrary fundamental units, is not only desirable but actually necessary. The simplicity of the dimensional relations obtained by using the practical system, and the ease with which such physical dimensions may be reproduced by simply remembering Ohm's law, Joule's law and Faraday's law of electromagnetic induction should recommend this system to the electrical profession.

PROBLEMS

1a. Memorize the names of the units, in the practical system, for all the physical quantities given in Table I.

1b. Look up in any handbook for electrical engineers the definitions of the fundamental units of length, mass, time, international ampere and international ohm.

1c. Look up the definitions of such magnetic units as the gilbert, the oersted, the maxwell (line) and the gauss. Correlate these units with the corresponding units in the practical system.

2a. Check the dimensional formulas, in the practical system, of all the units given in Table I.

2b. Given that the magnetic pull, F , of an electromagnet depends solely upon the area, A , of the air-gap, the magnetic permeability, μ , of the air, and the flux density, $\mathcal{B}$, in the air-gap. Determine how each of these factors enters into the equation for F .

Hint. — Using the practical system:

$$[W] = [FL] \quad \therefore [F] = [WL^{-1}] = [RI^2TL^{-1}]$$

CHAPTER V

THE CHARACTERISTICS OF AN ELECTRIC CIRCUIT

1. Introduction. — An isolated electric circuit may comprise resistance, inductance, or (and) capacitance. These properties of a circuit shall be called its characteristics. It is imperative that the student understand the physical nature of these characteristics before undertaking a study of alternating electric circuits. This chapter is therefore devoted to them.

2. Electric Resistance R. — With our present meager knowledge of the nature of electricity it is not possible to explain the physical nature of electric resistance. We can, however, define it through certain empirical relations. These are (a) Ohm's Law and (b) Joule's Law.

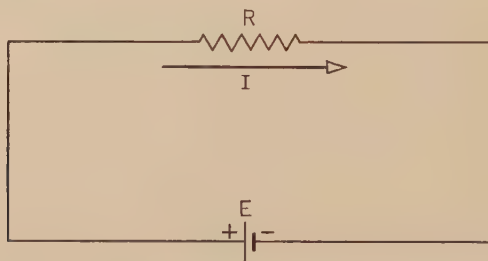


FIG. 1. A simple electric circuit obeying Ohm's law.

(a) *Ohm's Law.* — If an unvarying Emf. (E , Fig. 1) be impressed on a homogeneous metallic conductor and if the temperature of that conductor be held constant, then the ratio of the current, I , in the conductor, to the impressed Emf., E , is constant. This constant ratio may be expressed in two ways:

$$\left. \begin{aligned} R &= E/I & (a) \\ G &= I/E & (b) \end{aligned} \right\}^* \quad (1)$$

R is known as the resistance of the conductor and $G = 1/R$ is called its conductance.

Eqs. (1) give a mathematical statement of Ohm's Law and incidentally define resistance. The following limitations of Ohm's Law must be emphasized:

* The practical system of electrical units is adopted in this text.

(a) Ohm's Law holds true only for homogeneous metallic conductors. It fails to be applicable, for example, in the case of liquid and gaseous conductors, of conducting dielectric materials, or of two dissimilar metals joined or welded together.

(b) Ohm's Law applies only when the source of Emf. is constant or when no local Emfs. exist in the circuit.

(c) Ohm's Law is applicable only when the temperature of the conductor is constant. Experiment shows that the resistance of a metallic conductor generally increases with an increase in temperature. Thus, if R_t is the resistance at a temperature t° C. and if R_0 is the resistance at 0° C. then

$$R_t = R_0(1 + \alpha t) \quad (2)$$

where α is a positive constant which depends upon the metal and the units used and is known as the temperature coefficient of the metal.

(b) *Joule's Law.* — The flow of current through a resistance is always accompanied by heat. Joule established the experimental fact that the amount of energy, W , dissipated in a metallic wire is proportional to the product of the square of the current, I , and the time t during which that current flows. Moreover the coefficient of proportionality between W and I^2t is the resistance R of the circuit, or

$$W = RI^2t \quad (3)$$

We thus have a definition of resistance from an energy viewpoint. Resistance is that characteristic of a circuit which accounts for the existence of heat. An electric circuit is devoid of resistance when the flow of current is not accompanied by loss of energy through heat. Some metals at a temperature of zero degrees absolute offer practically no resistance to the flow of electricity.

Now, according to Ohm's Law, $E = RI$. Therefore Eq. (3) becomes:

$$W = (RI)It = EIt \quad (3')$$

which is the general expression for the energy supplied by a source of Emf. to a circuit obeying Ohm's Law.

(c) *Resistance as an Intrinsic Property of Metals.* — Although resistance is computed and defined through the current and voltage relations or through the energy dissipated in an electric circuit it must not be inferred that it is a function of either E , I or W . Resistance is independent of all. It is a function of the material of the conductor and its physical dimensions.

Experiment shows that the resistance of a straight conductor having a constant length h in the direction of current flow and a constant cross-

sectional area A perpendicular to the flow is:

$$\begin{aligned} R &= \rho(h/A) & (a) \\ \text{whence } G &= (1/R) = (1/\rho) (A/h) = \gamma(A/h) & (b) \end{aligned} \quad (4)$$

where ρ is known as the resistivity and $\gamma = (1/\rho)$ is called the conductivity of the metal. ρ and γ are defined as the resistance and conductance respectively of a cube having sides of unit length (a centimeter cube or an inch cube).*

The resistance of a conductor may be computed by the use of Eq. (4) only if that conductor satisfies the following conditions:

- (a) Its length in the direction of current flow must be constant.
- (b) Its cross-section must be uniform and constant.

When these two conditions are not satisfied there exist two alternative methods of determining resistance:

- (a) Experimentally through the use of Ohm's or Joule's Law.
- (b) Mathematically through the use of calculus.

The latter method is limited in its scope to conductors having shapes which can be expressed by simple mathematical relations. Sample problems (Nos. 2f and 2h) are solved to show how the resistance of a conductor, which does not comply with the limitations of Eq. (4), may be computed mathematically.

3. Inductance $\mathcal{L}$. — Inductance (sometimes referred to as the inertia of a circuit) is that characteristic of an electric circuit by virtue of which a sudden increase or decrease of the current is checked. Viewed from the standpoint of energy, inductance is that property which causes an electric circuit to store up energy while the current increases, and to deliver energy while the current decreases. We shall elucidate these two viewpoints of inductance.

(a) *The Magnetic Field Accompanying a Current and the Nature of Inductance.* — Experiment shows that a circuit carrying an electric current is encircled by a magnetic field in a plane perpendicular to the direction of current flow. The magnetic field may be imagined to consist of lines or tubes of force (Fig. 2) which form closed paths around the current. Since current is a directed quantity these tubes of force (Fig. 2) will assume reverse directions when they surround oppositely flowing currents. We shall assume that the direction of current flow and that of

* In defining resistivity and conductivity, distinction must be made between centimeter cube and cubic centimeter. The former specifies not only the volume but the dimensions as well. The latter gives the volume irrespective of dimensions. Resistivity is the resistance of a centimeter cube and not of a cubic centimeter. Indeed the resistivity of a cubic centimeter is an indefinite quantity.

the magnetic field correspond respectively to the direction of the progress of a right-handed screw and that of its rotation. Thus imagine a screw to be turned clockwise it will progress away from the reader. A counterclockwise rotation makes a screw progress toward the observer. Similarly if a current is directed into the plane of the paper (see $\times$, Fig. 2) then the flux surrounding it is clockwise; and vice versa. These facts are illustrated in Fig. 2 which shows cross-sections of two conductors carrying oppositely directed currents. The current flowing toward the reader is indicated by a dot. The current directed away from the reader is shown by a cross ($\times$). Circles representing the lines of force are drawn around the conductors. The direction of these lines of force follows the convention stated above.

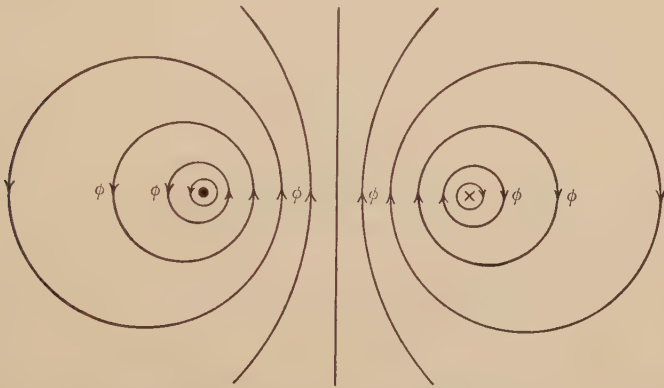


FIG. 2. Current flowing through a circuit and the flux surrounding it.

If the magnetic permeability of the circuit and that of the ambient medium are constant (that is if the phenomenon of magnetic saturation does not exist) then the flux linking with the circuit (Fig. 2) is proportional to the current, or:

$$\phi = (\mathcal{L}/N)i \quad (5)$$

where ϕ = the magnetic flux linking with the circuit.
 i = current in the circuit.

$(\mathcal{L}/N)$ = coefficient of proportionality between ϕ and i . The value of $(\mathcal{L}/N)$ depends upon the physical dimensions of the circuit and the units used.

Experiment further shows that, if the flux linking a circuit be changed, then an Emf. e_L is induced in the circuit which is proportional to the time-rate of change of flux. In other words:

$$e_L = -N(d\phi/dt) \quad (6)$$

where N is the number of turns in the circuit which are completely linked by the flux ϕ .

We thus have these two experimental facts:

- (1) In media of constant permeability a change in the current produces a proportional change in the flux encircling it. (Eq. 5).
- (2) A change in the flux linking with a circuit induces an Emf. in that circuit. (Eq. 6).

The inevitable conclusion from these two premises is that a change in the current flowing in a circuit produces an Emf. in that circuit. Experiment shows that such indeed is the case. Mathematically this deduction may be easily verified by substituting Eq. (5) in (6). This gives:

$$e_L = - \mathcal{L}(di/dt) \quad (7)$$

$\mathcal{L}$ is known as the inductance, self-inductance, or coefficient of self-induction of the circuit. It is the coefficient of proportionality between the voltage e_L induced in a circuit and the time rate of change of current in it. The symbol for inductance is a helix (see Fig. 3).

The negative sign in Eq. (7) indicates that the voltage induced in a circuit, due to a change in current, is such as to oppose the change. Thus if the current increases, then a voltage is induced which tends to check that increase; and vice versa. Lenz was the first to observe this fact. Its validity can be easily seen from the following reasoning.

Suppose that a continuous Emf. is suddenly impressed on a series circuit having resistance and inductance. The current, heretofore zero, will in time assume a certain definite steady value, I . In the interim the current must increase from zero to I . If each successive increase induces a voltage which tends to further increase the current, then, instead of assuming a finite value I , the current will, ultimately, become infinite — which is contrary to physical facts.

(b) *Inductance Defined in Terms of Flux Linkages.* — Solving Eq. (5) for $\mathcal{L}$ we have

$$\mathcal{L} = (N\phi)/i \quad (8)$$

Eq. (8) states that the inductance of a circuit is equal to the flux per unit current which links with the turns, N of the circuit. The product $(N\phi)$ is often called the flux linkages. Hence inductance may be defined as the flux linkages per unit current.

(c) *The Limitations of the Concept of Inductance.* — In defining inductance by Eq. (7) certain restrictions were imposed. These are given below:

(1) The concept of inductance applies only to conductors made of materials of constant permeability and surrounded by media of constant permeability. We cannot, for example, speak of the inductance of an iron conductor nor can we refer to the inductance of a copper wire wound around an iron core unless it is agreed that the term is to be used only for that part of the $\mathcal{B}$ - $\mathcal{H}$ curve where the permeability is constant. This is due to the fact that (Eq. 5) must hold true.

(2) The Emf. induced in a circuit by virtue of its inductance is invariably due to the change of current carried by that circuit itself and not due to any other cause. Thus, the Emf. induced in the armature winding of a generator due to its rotation in a magnetic field is not an Emf. of self-induction. This is why the term self-inductance is preferable. It is more descriptive of the phenomenon.

(3) Self-inductance disappears if, by any means whatever, the magnetic field, due to a current, is eliminated. Under these conditions a change in current does not produce a corresponding change in the flux because ϕ is non-existent. Although this is not fully realizable, an approach to it may be had if the circuit is wound non-inductively.

(d) *Inductance as a Storehouse of Electromagnetic Energy.* — From an energy viewpoint, inductance may be defined as that property of an electric circuit by virtue of which energy is stored in a circuit while the current increases and given up while the current decreases. We shall amplify this statement by describing an experiment and actually computing the stored energy.

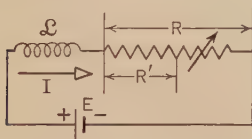


FIG. 3a. A circuit consisting of a variable R , a constant $\mathcal{L}$, and an Emf. E .

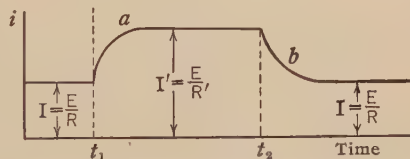


FIG. 3b. Curve showing change of current with time as R is suddenly varied.

Fig. 3a shows a circuit with a continuous Emf. source E , a variable resistance R which may be reduced to R' , and a constant inductance $\mathcal{L}$. Let the resistance be maintained at a value R and let steady conditions be established. The steady current, according to Ohm's Law is (see Fig. 3b):

$$I = (E/R) \quad (9)$$

Next let the resistance be suddenly reduced to R' at the instant t_1 . The final steady current is (see Fig. 3b):

$$I' = (E/R') \quad (10)$$

The current does not increase instantly from I to I' but rather rises gradually following curve a , Fig. 3b. During the time of increase a voltage of self-induction exists of such a direction as to oppose the impressed voltage. In other words one part of the impressed voltage is devoted to counterbalance the Emf. of self-induction and the other part is available to send current through the circuit. That part of the impressed voltage which overcomes e_L is:

$$e'_L = -e_L = \mathcal{L}(di/dt) \quad (11)$$

If the instantaneous value of the current is i , then the energy supplied by the Emf. source in time dt at an Emf. e'_L is:

$$dW = e'_L i dt = \mathcal{L} i di \quad (12)$$

and the total energy supplied by the source and stored in the circuit by virtue of its inductance is:

$$W = \int_I^{I'} \mathcal{L} i di = (1/2) \mathcal{L}(I'^2 - I^2) \quad (13)$$

It can be similarly shown that during the decrease of the current (see curve b , Fig. 3b) from I' to I the magnitude of the energy returned from the circuit to the continuous Emf. source is also expressed by Eq. (13)*

(e) *Inductance as an Intrinsic Property of a Circuit.* — The reader would ask whether $\mathcal{L}$ could possibly be computed from the dimensions of a circuit. The answer is yes only in very simple and special cases. It is outside the scope of this text to deal with this phase of inductance. It will therefore be assumed that the inductance of a circuit is either known or can be experimentally determined.

4. Capacitance. — When a dielectric separates two conductors we have a capacitor or condenser. The dielectric may be air or any other non-conducting material. The conductors which the dielectric separates may be two metallic plates, two wires, a wire and the earth, metallic spheres, or a combination of any two or more of the above mentioned conductors. The symbol for capacitance is two parallel lines (see d and f , Fig. 4a). For simplicity of treatment we shall assume that the dielectric separating the plates is perfect (for example, vacuum or air) and that the voltage is kept low enough to eliminate corona and other disturbing influences.

(a) *Relation between the Charge and the Charging Current.* — Fig. 4a shows a circuit consisting of a resistance R , a parallel plate capacitor C ,

* The expression for returned energy is the negative of Eq. (13). The negative sign simply means that the Emf. source is receiving rather than supplying energy.

a ballistic galvanometer G , an oscillograph O , a source of continuous Emf. E and a double throw switch S . Upon moving S to the point a , the Emf. E is impressed on the capacitor C . The ballistic galvanometer shows a deflection indicating that a quantity of electricity Q has passed from the source to the capacitor. This charge of electricity is nothing but a momentary current which continues to flow for a short time and then ceases altogether as shown by the curve i_c , Fig. 4b.

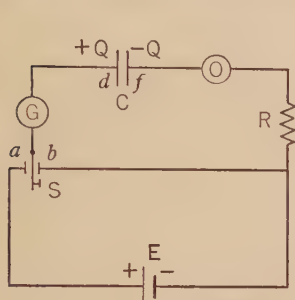


Fig. 4a

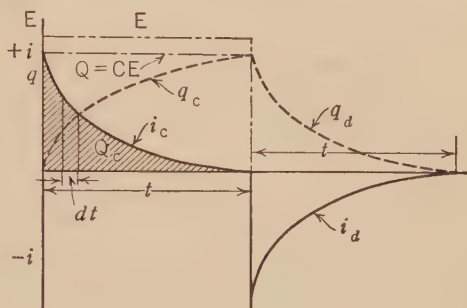


Fig. 4b

FIG. 4a. A circuit with R , C , and E .FIG. 4b. Curves of current and charge during the charge and discharge of C .

Now electric current may be regarded as consisting of a transfer of electricity along a conductor. Consequently the amount (or quantity) of electricity which passes through any cross-section of a conductor in time t is proportional to the magnitude of the current and the time of flow. Thus the readings of the ballistic galvanometer and the oscillograph may be coordinated through the following equations:

$$\left. \begin{aligned} Q &= \int_0^t i \, dt \quad (a) \\ i &= dq/dt \quad (b) \end{aligned} \right\} \quad (14)$$

or

Eq. (14a) states that the total charge acquired by the condenser is the area under the curve i_c , Fig. 4b. Eq. (14b) states that the magnitude of the current flowing in the circuit, Fig. 4a, is at every instant equal to the rate at which the quantity of electricity is carried along the wires. This is graphically the slope of the curve q_c , and is represented by the curve i_c (Fig. 4b).

(b) *The Charging Emf. is Equal to the Potential Across the Plates of a Charged Capacitor.*—Let the switch S be open after a time t , (Fig. 4b) when the current has become zero. No spark is observed when the switch is open because no current has been interrupted. Upon examining the plates d and f , it will be found that the plate d

has a positive charge $+Q$ and f has an equal negative charge $-Q$. Moreover, measurement of the potential difference E_c between the plates shows that the voltage is actually equal to the impressed Emf. E or

$$E = E_c \quad (15)$$

Under these conditions the capacitor is said to be fully charged.

(c) *Equality of the Charge and Discharge.* — Let the switch S be now moved to the point b . The capacitor begins to discharge through the resistance R . The galvanometer shows a deflection equal in magnitude to that previously indicated but opposite in direction. This indicates that the quantity of electricity Q_d obtained during the discharge is equal to the quantity Q_c supplied during the charge, or:

$$Q_c = Q_d \quad (16)$$

A capacitor may therefore be defined as a device for storing electricity — not in the form of electromagnetic energy, as in the case of inductance, but in its “raw state,” (the electric charge).

(d) *Relation between the Charging Emf. and the Charge.* — Experiment shows that the charge acquired by a capacitor whose conducting terminals are separated by a perfect dielectric is proportional to the Emf. across its plates. This proportionality between E_c and Q may be expressed in the following two ways:

$$\left. \begin{aligned} C &= Q/E_c & (a) \\ S &= E_c/Q & (b) \end{aligned} \right\} \quad (17)$$

C is known as the capacitance of a condenser (or capacitor) and $S = 1/C$ is known as its elastance. Eq. (17) is very similar to Eq. (1) and expresses what may be termed Ohm's law for the electrostatic circuit.

Eq. (17a) gives a definition of capacitance. It is simply the ratio Q/E_c .

(e) *Capacitance as an Intrinsic Property of a Condenser.* — The capacitance C is, for perfect dielectrics, independent of both Q and E and is simply a function of the dimensions and material of the capacitor. Experiment shows that, for capacitors made of plates having equal areas, placed parallel to each other, and separated by a homogeneous perfect dielectric of area A and thickness h , the capacitance C is:

$$C = K\kappa_a A/h^* \quad (18)$$

* Eq. (18) is true under the following limitations:

- (a) It neglects fringing of the electrostatic flux at the edges of the plates.
- (b) It assumes a perfect contact between the plates and the dielectric.
- (c) It assumes a non-anomalous perfect dielectric in which no hystero-viscosity or conduction effects can be observed.

where K = a constant known as the relative permittivity of the material. It is the ratio of the permittivity κ of a material to the permittivity κ_a of vacuum or:

$$K = \kappa / \kappa_a \quad (18a)$$

where κ_a = the permittivity of vacuum and may be taken as 8.842×10^{-14} farads per cm. cube.

A = area of one plate or of a cross-section of the dielectric in sq. cms.

h = distance between plates in cms.

(f) *Capacitance Viewed from an Energy Standpoint.* — The energy supplied to a capacitor is:

$$dW = e i dt = e dq \quad (19)$$

But by Eq. (17):

$$dq = C de \quad (20)$$

Therefore

$$W = \int_0^{E_c} C e de = (1/2) C E_c^2 = (1/2) C E^2 \quad (21)$$

whence

$$C = 2 W / E^2 \quad (22)$$

Thus the capacitance of a condenser is a measure of its capacity to store electrical energy at a given applied potential.

5. Conclusion. — The characteristics of an electric circuit may be either constant or variable. By the term *variable* is meant that these characteristics may assume arbitrary values from zero to infinity depending upon the conditions of the problem; for example, an open circuit may be considered as one having an infinite resistance.

Again, the characteristics may be concentrated as in the case of a laboratory circuit, or distributed as in the case of a transmission line.

Finally, these characteristics may be present either individually or collectively; that is, a circuit may consist of only one of these characteristics or it may consist of the following combinations of them:

- (a) R and $\mathcal{L}$.
- (b) R and C .
- (c) $\mathcal{L}$ and C .
- (d) R , $\mathcal{L}$, and C .

In the following pages we shall deal with all these types of circuits, arranging the work so that gradual progress is made from the simple to the relatively complicated circuits.

PROBLEMS

2a. Determine the value of the current that would flow through a resistance of 5 ohms when the applied Emf. is 10 volts.

2b. One method of measuring the unknown resistance of a circuit is to apply a continuous Emf. and take simultaneous voltage and current readings. Let these readings be $E = 15$ volts and $I = 10$ amp. What is the resistance of the circuit?

2c. What is the resistance of a copper rod 1 meter long and 0.015 cm. in diameter to axial current flow if the resistivity of copper is 1.724×10^{-6} ohm per centimeter cube?

2d. The resistivity given in problem 2c is for 20°C . What will be its value at 40°C . if the temperature coefficient α is 3.93×10^{-3} ?

2e. Compute the resistance of the rod specified in problem 2c for a temperature of 300°C .

2f. A hollow cylinder, made of a homogeneous metal, has internal and external radii a and b respectively and axial length h . If the resistivity of the metal is ρ , compute the resistance of the cylinder to radial flow:

Solution. Take an element of the cylinder at a radius x and let the thickness of the element be dx . Its resistance will then be:

$$dR = \rho dx / 2\pi xh$$

Since the various elements are in series the sum of the resistances of these elements is equal to the resistance of the cylinder. Or

$$R = (\rho/2\pi h) \int_a^b (dx/x) = (\rho/2\pi h) \ln (b/a)$$

2g. Compute the resistance of the cylinder, specified in problem 2f, to axial flow of current.

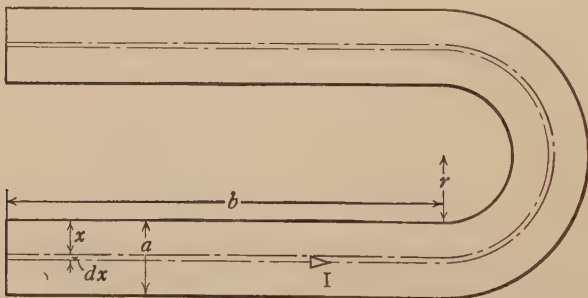


FIG. 5. Resistance of a conductor having variable length in the direction of current flow.

2h. A piece of steel is shaped like a horseshoe magnet and has the dimensions shown in Fig. 5. Its dimension perpendicular to the plane of the paper is h , and its conductivity is γ . Compute its resistance if the current follows paths similar to that indicated by the dotted line.

Solution. The conductance of an element at a distance x and of thickness dx is:

$$dG = \gamma h dx / [2b + \pi(r + x)]$$

Since the various elements are in parallel, the total conductance of the metal is equal to the sum of the conductances of the various elements, or:

$$G = (\gamma h/\pi) \int_0^a \frac{dx}{(2b/\pi) + r + x} = (\gamma h/\pi) \ln [1 + a/\{r + (2b/\pi)\}]$$

and

$$R = 1/G$$

2i. A spherical shell of inner and outer radii a and b respectively, has a resistivity ρ . What is the resistance of the shell to radial flow?

3a. Outline the various definitions of inductance given in the text and make clear to yourself just what the phenomenon is physically.

3b. A current is varied uniformly at the rate of 2 amp. per sec., giving a voltage reading of 5 millivolts. What is the inductance of the circuit?

3c. Determine the amount of energy stored in a reactance coil having an inductance of 15 mh. (15×10^{-3} henry) when a current of 50 amp. flows through the coil.

3d. If the current in problem 3c is increased from 50 to 80 amp. what additional energy would be stored?

3e. Assume the current in problem 3b to be decreased to 10 amp. Compute (a) the energy that would remain in storage and (b) the energy that would be given up. Add the two amounts and the total energy should check the answer to problem 3b.

3f. Devise some means of rendering a circuit non-inductive.

3g. Show that when the current, Figs. 3, decreases from I' to I , the energy delivered to the Emf. source is:

$$W' = (1/2) \mathcal{L} (I'^2 - I^2)$$

4a. A capacitor is charged at the rate of 10^{-6} amp. per sec. How much charge will it have at the end of one hour?

4b. The charging current delivered to a condenser is of the form:

$$i = 15 e^{-60t}$$

If the condenser is initially discharged, how much charge will it acquire in 1 second, 2 seconds, 10 seconds?

4c. The voltage across the terminals of a charged condenser is 100 volts. If the capacitance of the condenser is $1\mu f$ (10^{-6} farad) what is the charge on the condenser?

4d. What is the elastance of a condenser whose capacitance is $3\mu f$?

4e. The insulation in a parallel plate capacitor has a permittivity of $0.5 \times 10^{-6}\mu f$ per cm. cube. The cross-sectional area of the insulation is 100 sq. cm. and its thickness is 0.2 cm. Compute the capacitance of the condenser. What is the relative permittivity of the dielectric?

4f. What is the energy stored in the condenser of problem 4c when it is subjected to a continuous Emf. of 500 volts?

CHAPTER VI

SIMPLE ALTERNATING ELECTRIC CIRCUITS

1. Definition of Sine Emfs. — An Emf. whose instantaneous value is a sine function of time is called a sine Emf. or sine voltage. In view of the fact that the value of a sine function alternates between positive and negative maxima (see Fig. 1, Chapter III), sine Emfs. fall under the more general designation of Alternating Emfs.

2. Electromagnetic Induction. — Faraday discovered that when relative motion exists between a metallic conductor and a magnetic field an Emf. is induced in the conductor. The magnitude of this Emf. is:

$$e = \mathfrak{B}vD \quad (1)$$

where e = voltage induced between the terminals of the conductor.

$\mathfrak{B}$ = flux density of the magnetic field in a direction perpendicular to the axis of the conductor.

D = length of the conductor.

v = relative velocity of the conductor and the magnetic field.

Eq. (1) may be easily remembered as the $\mathfrak{B}vD$ Law. Its essential limitations are:

- (a) $\mathfrak{B}$, v , and D shall be directed along three mutually perpendicular axes as shown in Fig. 1.

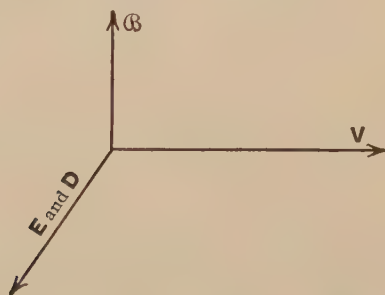


FIG. 1. Relation between the directions of the velocity, flux density and Emf. vectors.

- (b) $\mathfrak{B}$ shall be uniform along the entire length of the conductor but not necessarily uniform along the direction of v .

- (c) Since the velocity v is relative it is immaterial whether the conductor or the flux performs the motion. As a matter of fact,

in most commercial alternators the conductor is stationary while the field revolves. In continuous current machines, however, the flux is stationary and the conductor revolves.

3. Generation of Sine Emfs. — Sine voltages are produced by machines (called alternators) which utilize Faraday's Law of electromagnetic Induction. In order to make this statement more real, we shall briefly describe such a machine,

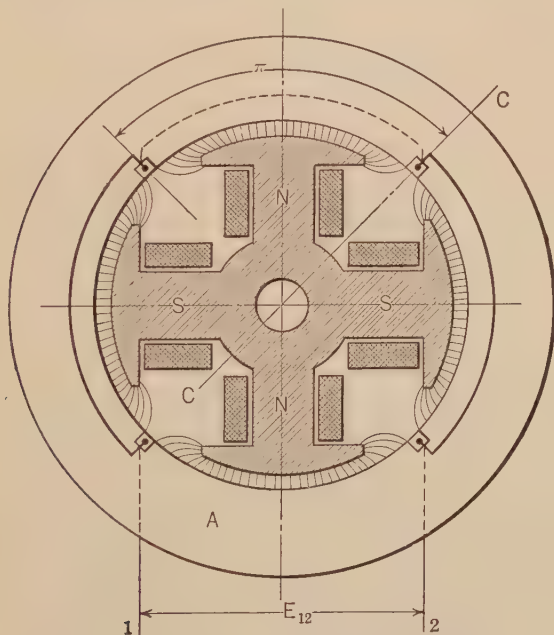


FIG. 2. Single-phase four-pole alternator.

Fig. 2 shows a cross-section of four magnetic poles marked *N* (north pole) and *S* (south pole). These poles are mounted on a shaft and are free to revolve inside an iron cylinder (*A*) known as the armature of the machine.

Four slots, 90° apart, are cut in the armature and two insulated coils, each consisting of one turn of copper wire, are placed in these slots and are connected in series.

Let the structure (Fig. 2) be cut radially along the line *CC* and the armature and poles be developed such that their relative positions be left unaltered. This gives Fig. 3.

The reader's attention is called to the following facts relative to Fig. 3:

(a) The flux emanating from a north pole is considered positive and

the flux entering into a south pole is negative. This is indicated, in Fig. 3, by drawing the flux density wave (sine wave with amplitude $\mathfrak{B}_m$) such that its positive values occur opposite a north pole and its negative values opposite a south pole.

(b) The flux density curve is assumed to be a sine function of space along the air-gap. Therefore its positive maximum occurs opposite the midpoint of a north pole and its negative amplitude is opposite the middle of a south pole. The mathematical expression for this wave (see Fig. 3) is:

$$\mathfrak{B} = \mathfrak{B}_m \sin (x + \alpha) \quad (2)$$

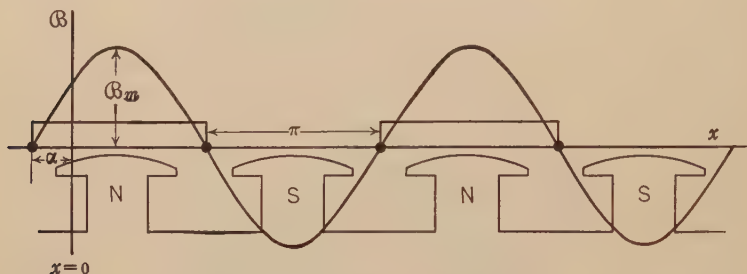


FIG. 3. Developed alternator.

(c) From remark (b) it follows that a pole pitch (the distance from one point on a north pole (Fig. 3) to a similar point on an adjacent south pole) is equal to π radians measured along the abscissa of the flux wave.*

Now let v be the relative velocity of the pole structure and D the length per conductor. Then the Emf. between the terminals of each conductor is, according to Eq. (1):

$$e = \mathfrak{B}_m v D \sin (x + \alpha) = E_m \sin (x + \alpha) \quad (3)$$

where

$$E_m = \mathfrak{B}_m v D \quad (4)$$

Let the field rotate through an electrical angle x in time t and let its velocity be such that it takes T seconds to move over a distance of 2π electrical radians. Then:

$$\begin{aligned} (x/2\pi) &= (t/T) \\ x &= (2\pi t/T) = 2\pi f t = \omega t \end{aligned} \quad (5)$$

* When distance along the periphery of the armature is expressed in radians, measured along the abscissa of the flux curve (Fig. 3), the term *electrical radians* is used. Distinction should be clearly drawn between an *electrical radian* and a *mechanical radian*. Thus there always exist π *electrical radians* (Figs. 2 and 3) between any two corresponding points of a north pole and an adjacent south pole, irrespective of the number of poles in a machine. On the other hand, considering that there are 2π *mechanical radians* in a circle, then the number of *mechanical radians* between any two adjacent pole centers is $2\pi/p$.

Substituting this value of x in Eq. (3) we have:

$$\begin{aligned} e &= E_m \sin (\omega t + \alpha) = E_m \sin (2 \pi f t + \alpha) \\ &= E_m \sin [(2 \pi t / T) + \alpha] \end{aligned} \quad (6)$$

We thus have a sine voltage of amplitude E_m and frequency f .* The effect of impressing such a voltage on various characteristics will now be considered.

4. Electric Circuits and Current and Emf. Notations. — (a) *Definition.* — The term electric circuit is used in this text to define a combination of one or more Emf. sources and one or more circuit-characteristics, so connected or coupled that current flows through the combination. Fig. 4 shows a simple electric circuit consisting of a sine Emf. E_{12} , a resistance R , and an inductance $\mathcal{L}$.

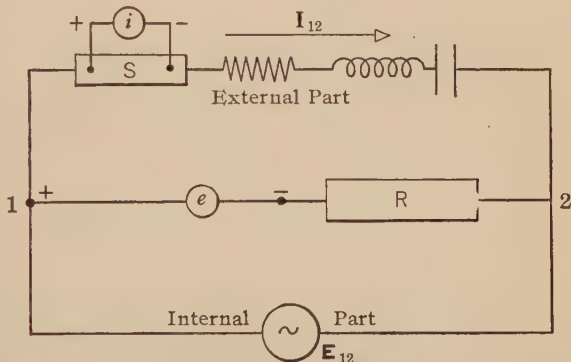


FIG. 4. Internal and external parts of an electric circuit.

(b) *Internal and External Parts of a Circuit.* — Designate by 1 and 2 the terminals of the Emf. source (Fig. 4). Evidently there are two paths that would be traced in passing over the circuit from point 1 to point 2. Path 1 passes through the Emf. source but not through the characteristics, whereas path 2 traverses the characteristics but not the source. We shall call these two paths the internal and external parts of the circuit respectively.

(c) *Oscillograph† Connections and Current and Emf. Notations.* — Let a two-element oscillograph be connected as shown in Fig. 4. Call its current and voltage elements i and e respectively. If the positive terminal of the voltage element be connected to point 1 and the negative

* It is desirable to develop a relation between the frequency of the Emf. generated and the number of revolutions that the pole structure makes. Consider an alternator having p poles and making S revolutions per second. Each conductor passes through $(p/2)$ cycles of flux per revolution because it takes two poles to make a complete cycle of flux (see Fig. 3). Thus, in one second it passes through $(pS/2)$ cycles. Consequently $f = (pS/2)$ cycles per second.

† The student is here urged to study the theory, construction, and operation of at least one type of oscillograph. This instrument is referred to rather frequently in the study of the physical phenomena prevailing in electric circuits.

terminal to point 2, then a sine wave would be recorded which we shall represent by the vector $\mathbf{E}_{12}$. On the other hand, if the terminals are reversed, then a wave would be obtained which is displaced from the first by 180° and which shall be represented by the vector $\mathbf{E}_{21}$.

Since the two voltage waves, so recorded, are 180° out of phase, the vectors representing them will be mutually negative or:

$$\mathbf{E}_{12} = -\mathbf{E}_{21} \quad (7)$$

The double-subscript notation for voltage vectors will be used throughout this text and will have two meanings:

- (1) The first and second subscript represent respectively the points to which the positive and negative terminals of an oscillograph are connected.
- (2) The first and second subscripts stand respectively for the origin and terminus of a voltage vector. Thus $\mathbf{E}_{12}$ represents a vector having an origin at 1 and a terminus at 2.

Now in order to obtain consistent phase relations the current element of the oscillograph must be so connected (see Fig. 4) that the positive terminal of the current element is approached first in tracing over the *external* part of the circuit from the positive to the negative terminal of the voltage element. The oscillograph of the current so obtained shall be represented by the vector $\mathbf{I}_{12}$ or $\mathbf{I}_{21}$, depending upon whether the voltage vector is $\mathbf{E}_{12}$ or $\mathbf{E}_{21}$, respectively.

The double-subscript notation for current vectors will be conserved throughout the text and will have the two meanings assigned to it in connection with Emf. vectors.

In general, a voltage $\mathbf{E}_{12}$ causes a positive current $\mathbf{I}_{12}$ to flow *in the external part of the circuit* from point 1 to point 2, and an Emf. $\mathbf{E}_{21}$ causes an external current $\mathbf{I}_{21}$. This system of notations is easy to remember because the sequence of subscripts for the voltage and *external current* is the same.

5. Circuits Consisting of a Non-Inductive Resistance R and a Sine Emf. — (a) *Physical Considerations.* — Let a circuit (Fig. 5a) consist of a non-inductive resistance R and a source of sine voltage $\mathbf{E}_{12}$. The current and voltage waves such as would be recorded by an oscillograph are given in Fig. (5c) which shows that:

1. Both waves are sinusoidal and have identical frequencies.*
2. The waves are in phase.

* The identity of their frequencies may be concluded from the fact that the length on the abscissa scale, representing one period, is the same for both waves.

Now let an ammeter and a voltmeter be connected in the circuit, Fig. 5a. The effective values of the current and voltage, as read by these instruments, will be such that:

$$E = RI \quad \text{or} \quad I = GE \quad (8)$$

These are the physical facts. Let us seek a mathematical interpretation of them.

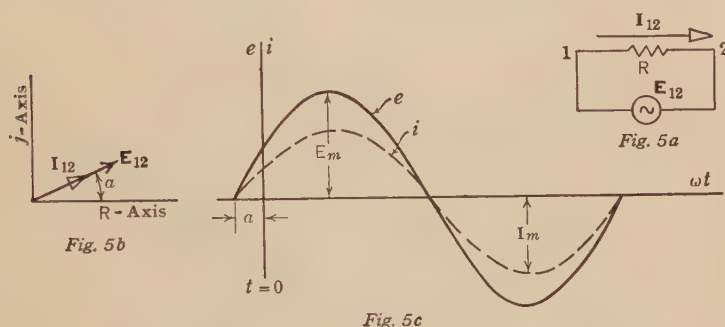


FIG. 5a. A circuit consisting of a resistance R and a sine Emf. E_{12} .

FIG. 5b. Relation between the voltage and current vectors in the circuit 5a.

FIG. 5c. Sine waves of Emf. and current in the circuit Fig. 5a.

(b) *Instantaneous and Effective Value Relations.*—Let the sine voltage impressed on the circuit (Fig. 5a) be:

$$e = E_m \sin(\omega t + \alpha) \quad (9)$$

Since Ohm's Law must hold true for every instant we have:

$$e = Ri^* \quad (10)$$

Substituting for e its value from Eq. (9) we have:

$$\left. \begin{aligned} i_r &= (E_m/R) \sin(\omega t + \alpha) & (a) \\ &= I_m \sin(\omega t + \alpha) & (b) \end{aligned} \right\} \quad (11)$$

$$\text{where} \quad I_m = (E_m/R) = GE_m \quad (12)$$

Dividing both sides of Eq. (12) by $\sqrt{2}$ and remembering that $I = I_m/\sqrt{2}$ and $E = E_m/\sqrt{2}$ (see Eq. 10, Chapter III) we obtain:

$$E = RI \quad \text{or} \quad I = GE \quad (8')$$

Eqs. (11) and (8') offer a complete explanation of the physical facts cited above. Thus Eq. (11) shows that the current is a sine wave in phase with the voltage wave and Eq. (8') is identical with Eq. (8).

* The term Ri shall be hereafter called the potential drop across the resistance.

(c) *Vector Relations and Complex Expressions.*— Let an arbitrary instant of time $t = 0$ be selected, as in Fig. 5c. Then the sine voltage and current may be represented by the stationary vectors shown in Fig. (5b).

The complex expressions for these vectors are:

$$\begin{aligned} \mathbf{E}_{12} &= E(\cos \alpha + j \sin \alpha) = E\epsilon^{j\alpha} & (a) \\ \mathbf{I}_{12} &= I(\cos \alpha + j \sin \alpha) = I\epsilon^{j\alpha} & (b) \end{aligned} \quad (13)$$

$$\text{But} \quad R = \mathbf{E}_{12}/\mathbf{I}_{12} = (E\epsilon^{j\alpha}/I\epsilon^{j\alpha}) = E/I \quad (14)$$

Resistance can therefore be considered as a real quantity in dealing with electric circuits.

6. Circuits Consisting of an Inductance $\mathcal{L}$ and a Sine Emf.— (a) *Physical Considerations.*— Let a circuit, Fig. 6a, consist of an inductance $\mathcal{L}$ and a source of sine voltage $\mathbf{E}_{12}$. The waves of the current and voltage that would be recorded by an oscillograph are given in Fig. 6c. An inspection of the oscillogram shows that:

- (a) Both waves are sinusoidal and have identical frequencies.
- (b) The voltage leads the current by $\pi/2$ radians.

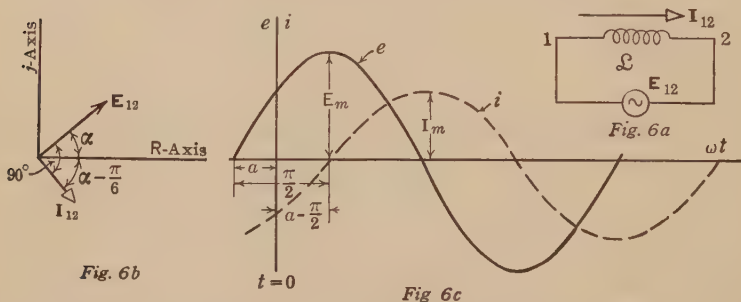


FIG. 6a. A circuit consisting of an inductance $\mathcal{L}$ and a sine Emf. $\mathbf{E}_{12}$.

FIG. 6b. Relation between the voltage and current vectors in the circuit Fig. 6a.

FIG. 6c. Sine waves of Emf. and current in the circuit Fig. 6a.

An ammeter and voltmeter connected in the circuit, Fig. 6a, give effective values of the current and voltage which are so related that:

$$E = X_L I \quad \text{or} \quad I = B_L E \quad (15)$$

where X_L and B_L are coefficients of proportionality between E and I , X_L is referred to as the *inductive reactance* of the circuit, and $B_L = 1/X_L$ is termed the *inductive susceptance*. If X_L is measured in ohms, then B_L would be expressed in mhos.*

* The terms X_L and B_L correspond respectively to R and G in a resistance circuit. The use of only one of them is entirely possible. But it is more convenient to retain both.

These are the physical facts. A mathematical interpretation of these facts will be now given.

(b) *Instantaneous and Effective Value Relations.* — Let the sine current flowing through the circuit (Fig. 6a) be:

$$i = I_m \sin [\omega t + \alpha - (\pi/2)] \quad (16)$$

Since the current represented in Eq. (16) varies with time, there will be induced in the circuit an Emf. which (according to Eq. (11) Chapter V) is:

$$\begin{aligned} e_L &= -\mathcal{L} di/dt^* \\ &= -\mathcal{L}\omega I_m \cos [\omega t + \alpha - (\pi/2)] \end{aligned} \quad (17)$$

Now the impressed voltage required to overcome the induced voltage will be equal to e_L but opposite in sign or:

$$\begin{aligned} e &= -e_L = (\mathcal{L}\omega I_m) \cos [\omega t + \alpha - (\pi/2)] & (a) \} \\ &= E_m \sin (\omega t + \alpha) & (b) \} \end{aligned} \quad (18)$$

$$\begin{aligned} \text{where} \quad E_m &= \mathcal{L}\omega I_m = (2\pi f\mathcal{L}) I_m = X_L I_m & (a) \} \\ \text{or} \quad I_m &= B_L E_m = (1/\omega\mathcal{L}) E_m & (b) \} \end{aligned} \quad (19)$$

$$\begin{aligned} \text{and} \quad X_L &= 2\pi f\mathcal{L} = \omega\mathcal{L} & (a) \} \\ B_L &= 1/X_L = (1/\omega\mathcal{L}) = (1/2\pi f\mathcal{L}) & (b) \} \end{aligned} \quad (20)$$

Dividing both sides of Eq. (19) by $\sqrt{2}$ and remembering that $E = E_m/\sqrt{2}$ and $I = I_m/\sqrt{2}$ we have:

$$E = X_L I \quad \text{or} \quad I = B_L E \quad (21)$$

In Eqs. (18b) and (21) we find a mathematical interpretation of the physical facts cited above. Thus Eq. (18b) states that the impressed voltage is a sine wave having the frequency of the current wave and leading the latter by 90° . Again Eq. (21) is identical with Eq. (15). Moreover Eq. (20) gives the value of the inductive reactance X_L in terms of the inductance of the circuit and the frequency of the source.

(c) *Vector Relations and Complex Expressions.* — Referring to Fig. 6c, let an arbitrary instant of time $t = 0$ be so selected that the current wave has its zero value ($\alpha - \pi/2$) radians after the instant $t = 0$; that is, let Eq. (16) be the analytical expression for the current wave. Then the current and voltage vectors will be as shown in Fig. (6b), in which the voltage $\mathbf{E}_{12}$ leads the current $\mathbf{I}_{12}$ by 90° . The complex expressions for these vectors is:

$$\begin{aligned} \mathbf{I}_{12} &= I e^{j(\alpha - \pi/2)} & (a) \} \\ \mathbf{E}_{12} &= E e^{j\alpha} & (b) \} \end{aligned} \quad (22)$$

* $e_L = -\mathcal{L} di/dt$ shall be hereafter termed the potential drop across the inductance.

Dividing Eq. (22b) by Eq. (22a) we have

$$\left. \begin{aligned} X_L &= E e^{j\alpha} / I e^{j(\alpha - \pi/2)} = jE/I = jX_L = j(2\pi f\mathcal{L}) & (a) \\ \text{whence } B_L &= 1/jX_L = -j(1/2\pi f\mathcal{L}) & (b) \end{aligned} \right\} \quad (23)$$

Eq. (23) shows that the inductive reactance X_L is a positive j -quantity whereas the inductive susceptance B_L is a negative j -quantity.

7. Circuits Consisting of a Capacitance C and a Sine Emf. — (a) *Physical Considerations.* — The circuit (Fig. 7a) consists of a capacitance C subjected to a sine voltage E_{12} . The form of the oscillogram which would be obtained for the current and voltage in such a circuit is given in Fig. (7c). Note that:

- (a) Both waves are sinusoidal and have identical frequencies.
- (b) The current wave leads the voltage wave by $\pi/2$ radians or 90° .

If an ammeter and voltmeter are properly connected in the circuit, Fig. (7a), the effective values of the current for different values of the voltage, will be such that:

$$E = X_c I \quad \text{or} \quad I = B_c E \quad (24)$$

where X_c and $B_c = (1/X_c)$ are coefficients of proportionality between E and I . X_c is termed the *capacitive reactance* and B_c the *capacitive susceptance* of the circuit.

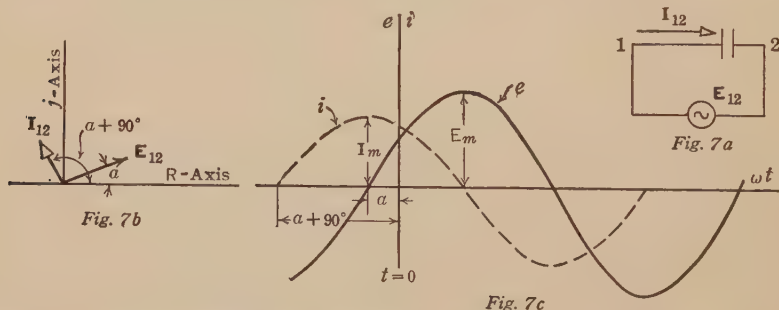


FIG. 7a. A circuit consisting of a capacitance C and a sine Emf. E_{12} .

FIG. 7b. Relation between the voltage and current vectors in the circuit Fig. 7a.

FIG. 7c. Sine waves of Emf. and current in the circuit Fig. 7a.

(b) *Instantaneous and Effective Value Relations.* — When a condenser (C , Fig. 7a) is subjected to an electric Emf. e , a charge q accumulates on the plates. The relation, at any instant, between the value of the charge q and the instantaneous value e of the applied Emf. is:

$$q = Ce^* \quad (25)$$

* Hereafter $e = q/C$ shall be called the potential drop across the capacitance.

Now if e is a sine function of time, then its instantaneous value may be expressed by Eq. (9). Substituting Eq. (9) in Eq. (25) we have:

$$q = CE_m \sin(\omega t + \alpha) \quad (26)$$

The instantaneous current flowing in the circuit (see Eq. (14b), Chapter V) is:

$$\begin{aligned} i = dq/dt &= (C\omega)E_m \cos(\omega t + \alpha) & (a) \\ &= I_m \sin(\omega t + \alpha + 90^\circ) & (b) \end{aligned} \quad (27)$$

where

$$I_m = C\omega E_m = (2\pi fC)E_m = B_c E_m = E_m/X_c \quad (28)$$

$$B_c = 2\pi fC \quad \text{and} \quad X_c = (1/B_c) = (1/2\pi fC) \quad (29)$$

Dividing both sides of Eq. (28) by $\sqrt{2}$ and using the fact that $E = E_m/\sqrt{2}$ and $I = I_m/\sqrt{2}$ we have:

$$E = X_c I \quad \text{or} \quad I = B_c E \quad (30)$$

Eqs. (27) and (30) give a complete interpretation of the physical facts cited above. Thus Eq. (27b) shows that the current in a capacitance circuit is a sine wave of the same frequency as the impressed voltage. Moreover, the current leads the voltage by 90° . Again Eq. (30) is identical with Eq. (24). Finally, Eq. (29) gives the value of the capacitive reactance X_c or capacitive susceptance B_c in terms of the frequency of the supply and the capacitance of the circuit.

(c) *Vector Relations and Complex Expressions.* — Referring to Fig. (7c), let an arbitrary instant of time $t = 0$ be so selected that Eq. (9) is the analytical expression for the voltage wave. Then the current and voltage vectors will be as shown in Fig. (7b). The complex expressions for these vectors are:

$$\begin{aligned} \mathbf{E}_{12} &= E\epsilon^{j\alpha} & (a) \\ \mathbf{I}_{12} &= I\epsilon^{j(\alpha+\pi/2)} & (b) \end{aligned} \quad (31)$$

Dividing Eq. (31a) by Eq. (31b) we have:

$$\begin{aligned} X_c = (\mathbf{E}_{12}/\mathbf{I}_{12}) &= E\epsilon^{j\alpha}/I\epsilon^{j(\alpha+\pi/2)}(E/I)\epsilon^{-j\pi/2} = -j(1/\omega C) \\ B_c &= 1/X_c = j\omega C \end{aligned} \quad (32)$$

Eq. (32) shows that the capacitive reactance X_c is a negative j -quantity whereas the capacitive susceptance B_c is a positive j -quantity — exactly the reverse of X_L and B_L .

8. *Comparison of Eqs. (20) and (29).* — The values of X_L and X_c as expressed by Eqs. (20) and (29) are:

$$X_L = 2\pi fL = \omega L \quad (20)$$

$$X_c = (1/2\pi fC) = (1/\omega C) \quad (29)$$

It will be observed that whereas the inductive reactance X_L is directly proportional to the frequency of the applied Emf. the capacitive reactance X_C is inversely proportional to the frequency. This means that a circuit consisting of an inductance offers greater opposition to the flow of current as the frequency increases. On the other hand in the case of a capacitance the opposition to the flow of current diminishes as the frequency increases.

With the inverse relations (B_L and B_C) the reverse holds true, that is, B_L decreases and B_C increases with an increase in frequency.

The significance of these facts is discussed in various places throughout the text, for example, in connection with resonance and equivalence of circuits.

PROBLEMS

1a. Give the instantaneous value expression for a sine Emf. having an amplitude E_m , a frequency f , and an initial angle $\alpha = 35^\circ$.

1b. Give the instantaneous value expression of a cosine Emf. having an amplitude E_m , an angular velocity ω , and an initial angle $\alpha = -25^\circ$.

1c. Convert the cosine function specified in problem 1b into a sine function.

2a. A conductor having a length of 50 cm. passes through a uniform magnetic field at a velocity of 50 meters per second. If the flux density of the field is 10^{-5} webers per sq. cm., what is the voltage induced between the terminals of the conductor?

2b. The original Faraday's dynamo consisted of a flat metal disk rotated between the poles of an electromagnet. Assume that the field of the magnet has a uniform flux density of 5×10^{-4} webers per sq. cm. If the radius of the disk is 10 cm. and its speed is 300 r.p.m., what is the voltage induced between the center of the disk and its periphery? (*Hint.* — Compute the voltage induced in an element dx at a distance x from the center of the disk. Then, through integration, add up the voltages induced in the various elements.)

3a. What are the period T and frequency f of a 2-pole turbo-alternator running at a speed of 3600 r.p.m.?

3b. The frequency of a certain Emf. source is 60 cycles. If the machine has 6 poles, at what speed is it operated?

4a. Draw two circuits each consisting of a resistance R and a sine source of Emf. Call a and b the points connecting the terminals of the resistance to those of the source. Now connect a two-element oscillograph in each of the circuits to give respectively:

(1) I_{ba} and E_{ba} .

(2) I_{ab} and E_{ab} .

4b. Sketch the sine waves that would be recorded by the voltage elements in circuits 1 and 2 of problem 4a. What do you note about these sine waves?

4c. Draw the vectors E_{ba} and E_{ab} representing the sine waves obtained in problem

4b. Are these vectors so related that $E_{ba} = -E_{ab}$?

5a. A sine voltage $e = 150 \sin (120 \pi t - 25^\circ)$ is impressed on a resistance of 15 ohms.

- Write the analytical expression for the instantaneous value of the current which flows in the resistance.
- Draw to scale the sine waves for the voltage and current, showing amplitudes, phase position, and the instant $t = 0$.
- Draw to scale a vector diagram of the voltage and current.
- Express the voltage and current vectors in the following complex forms:
 - Orthogonal.
 - Trigonometric
 - Exponential.

5b. A sine voltage was impressed on a circuit of pure resistance and readings were taken which gave $E = 110$ volts and $I = 20$ amp. What is the resistance of the circuit? What is its conductance?

5c. A two-element oscillograph, connected in a resistance circuit, shows the record of the sine voltage and current given in Fig. 8. Assuming that the time $t = 0$ is arbitrarily selected as indicated:

- Write the analytical expressions for the instantaneous values e and i .
- Draw to scale a vector diagram showing $\mathbf{E}_{12}$ and $\mathbf{I}_{12}$.
- Express the voltage and current vectors in the following complex forms:
 - Orthogonal.
 - Circular.
 - Exponential.
- Find the resistance R and conductance G of the circuit.

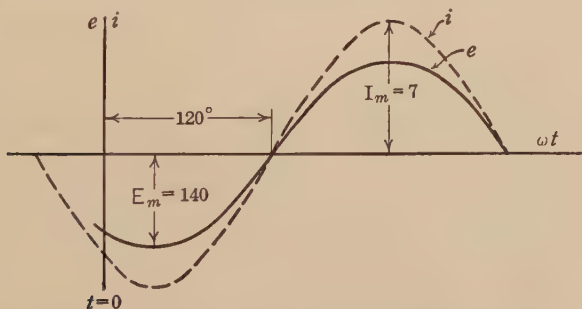


FIG. 8. Sine waves of e and i in a resistance circuit.

6a. The sine voltage specified in problem 5a was impressed on an inductance of 20 mh. (millihenry).

- Determine the numerical value of the inductive reactance and inductive susceptance of the circuit.
- Write the expression for the instantaneous current which flows in the circuit.
- Draw to scale the sine waves for the voltage and current, showing amplitudes, phase positions, and the instant $t = 0$.
- Draw to scale the current and voltage vectors showing phase angles and the axis of reference.
- Express the voltage and current vectors in three complex forms.

6b. A two-element oscillograph, connected in an inductance circuit, shows the record of the sine voltage and current given in Fig. 9. Assuming $t = 0$ to be arbitrarily selected as shown:

- Write the expressions for the instantaneous values e and i .
- Draw to scale a vector diagram showing E_{12} and I_{12} .
- Express the voltage and current vectors in three complex forms.
- Find the numerical values of the inductive reactance X_L and the inductive susceptance B_L .
- Express X_L and B_L as imaginary quantities.
- If the frequency of the source is 5000 cycles per second, what is the inductance of the circuit?

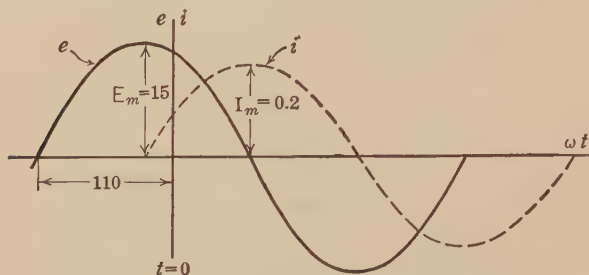


FIG. 9. Sine waves of e and i in an inductance circuit.

7a. A sine voltage $e = 10 \sin(6000\pi + 10^\circ)$ is applied to a $5 \mu f$ (5×10^{-6} Farad.) condenser.

- Find the numerical values of the capacitive reactance X_c and the capacitive susceptance B_c .
- Write the expression for the instantaneous current which flows in the circuit.
- Draw to scale the sine waves for voltage and current, showing amplitudes, phase positions, and the instant $t = 0$.
- Draw to scale the current and voltage vectors showing phase angles and the axis of reference.
- Express the voltage and current vectors in three complex forms.

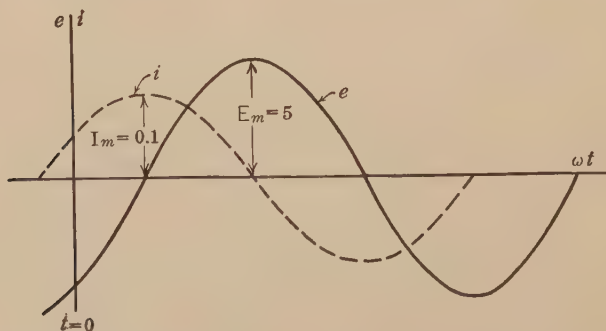


FIG. 10. Sine waves of e and i in a capacitance circuit.

7b. A two-element oscillograph connected in a capacitive circuit shows the record of the sine voltage and current given in Fig. 10. Assuming $t = 0$ to be arbitrarily selected as indicated:

- (a) Write the expressions for the instantaneous values e and i .
- (b) Draw to scale the vector diagrams of $\mathbf{E}_{12}$ and $\mathbf{I}_{12}$.
- (c) Express the voltage and current vectors in three complex forms.
- (d) Find the numerical values of the capacitive reactance X_c and the capacitive susceptance B_c .
- (e) Express X_c and B_c as imaginary quantities.

CHAPTER VII

SERIES CIRCUITS WITH CONSTANT CONCENTRATED CHARACTERISTICS

1. Definition. — The characteristics of a circuit are said to be in series when the same current flows through them. Thus in Fig. 1a the resistance R and inductance $\mathcal{L}$ are in series because the same current I_{12} flows through both.

2. Kirchhoff's Second or Emf. Law. — Kirchhoff's Emf. Law for alternating electric circuits may be stated thus:

In any closed circuit the algebraic sum of the instantaneous values of the Emfs. and the potential drops is zero. This law is evident from the simple fact that in going around a closed circuit one is bound to terminate at the point where he started. If the sum, at every instant, of the Emfs. and the potential drops were other than zero it would mean that the same point has two different potentials. This is physically impossible because a point can have but one electric potential.

Now in a series circuit with a single Emf. source (Fig. 1a) the flow of current in the external part of the circuit (through the characteristics) is such as to cause potential drops or to induce local Emfs. which, at every instant, are opposite in direction to the impressed Emf. Consequently the above law may be stated in the following form:

The instantaneous value of the Emf. impressed on a series circuit is equal to the sum of the instantaneous values of the potential drops.

As this chapter deals exclusively with circuits having a single Emf. source this statement of the law is deemed sufficient. The application of Kirchhoff's Emf. law to more complicated circuits requires further comments, which will be given in their proper place.

3. A Series Circuit Consisting of Resistance, Inductance, and a Sine Emf. — (a) *Physical Considerations.* — A series circuit, consisting of a resistance, R , an inductance, $\mathcal{L}$, and a sine Emf., E_{12} , is shown in Fig. 1a. Let a four-element oscillograph be properly connected to this circuit so as to give:

- (a) The applied Emf.
- (b) The circuit current.
- (c) The drop across the resistance.
- (d) The drop across the inductance.

The waves so obtained are similar to those given in Fig. 1c. Note that:

- All the waves are sinusoidal and have identical frequencies.
- The current wave, i , lags the impressed voltage wave, e , by an angle $\phi < 90^\circ$.
- The voltage drop, e_r , across the resistance is in phase with the current.
- The voltage drop e_L across the inductive reactance leads the current by 90° .

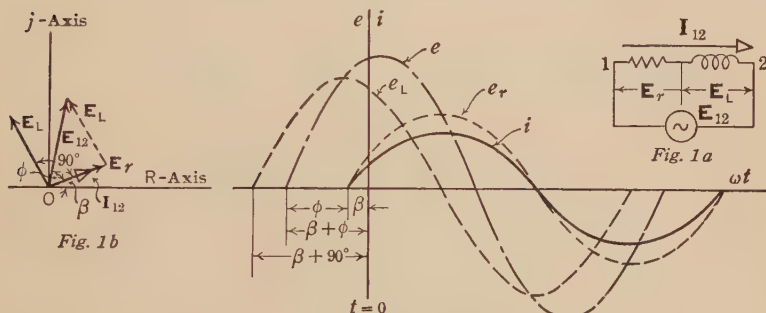


FIG. 1a. A series circuit consisting of R , $\mathcal{L}$, and an Emf. E_{12}

FIG. 1b. Vector diagram for the series circuit Fig. 1a.

FIG. 1c. Sine waves in the series circuit Fig. 1a.

Let an ammeter and a voltmeter be connected in the circuit, and simultaneous readings of voltage and current be taken for different values of E , the following relation will be found to hold true:

$$E = Z_L I \quad (1)$$

where Z_L is a coefficient of proportionality between the effective values E and I . Z_L will be hereafter referred to as the *impedance* of the series circuit. If E is measured in volts, and I in amperes then Z is expressed in ohms. These are the physical facts which we shall endeavor to explain mathematically.

(b) *Instantaneous and Effective Value Relations.*—Let the instantaneous value of the sine current flowing through the circuit (Fig. 1a) be:

$$i = I_m \sin(\omega t + \beta) \quad (2)$$

The flow of current through such a circuit results in two voltage drops, the one across the resistance and the other across the inductance. Hence by Kirchhoff's Emf. law:

$$e = Ri + \mathcal{L}(di/dt) \quad (3)$$

Substituting Eq. (2) in Eq. (3) we have:

$$\begin{aligned} e &= RI_m \sin(\omega t + \beta) + (\omega \mathcal{L} I_m) \cos(\omega t + \beta) & (a) \\ &= E_{mr} \sin(\omega t + \beta) + E_{mL} \sin(\omega t + \beta + 90^\circ) & (b) \end{aligned} \quad (4)$$

where

$$\begin{aligned} E_{mr} &= RI_m & (a) \\ E_{mL} &= \omega \mathcal{L} I_m = X_L I_m & (b) \end{aligned} \quad (5)$$

E_{mr} is the amplitude of the sine-voltage drop across the resistance and E_{mL} is the amplitude of the sine-voltage drop across the inductive reactance.

But the sum of two sine waves of the same frequency is a sine wave. Therefore, applying Eqs. (16), (17) and (19) of Chapter III to Eq. (4b); we obtain:

$$e = E_m \sin(\omega t + \beta + \phi) \quad (6)$$

where

$$\begin{aligned} E_m &= [E_{mr}^2 + E_{mL}^2]^{\frac{1}{2}} & (a) \\ \tan \phi &= E_{mL}/E_{mr} & (b) \end{aligned} \quad (7)$$

Substituting in Eqs. (7) the values of E_{mr} and E_{mL} from Eqs. (5) we have:

$$\begin{aligned} E_m &= I_m \sqrt{R^2 + X_L^2} = Z_L I_m & (a) \\ \phi &= \tan^{-1}(X_L/R) & (b) \end{aligned} \quad (8)$$

where the impedance

$$Z_L = E_m/I_m = \sqrt{R^2 + X_L^2} \quad (9)$$

Dividing both sides of Eq. (8a) by $\sqrt{2}$ and remembering that $E = E_m/\sqrt{2}$ and $I = I_m/\sqrt{2}$ we have:

$$E = Z_L I \quad (10)$$

The foregoing theory offers a complete interpretation of the physical facts cited in Art. 3a. Thus:

- (a) The fact that all the waves are sinusoidal is shown by Eqs. (2), (4), and (6).
- (b) A comparison of Eqs. (2) and (6) shows that the impressed voltage leads the current by an angle ϕ which, according to Eq. (8b), is less than 90° .
- (c) Upon comparing Eq. (2) with Eqs. (4) we conclude that the voltage drop across the resistance is in phase with the current while that across the inductance leads the current by 90° .

Finally Eqs. (10) and (1) are identical. Moreover, Eq. (9) gives a numerical value of the coefficient of proportionality Z_L in terms of the resistance and inductive reactance of the circuit.

(c) *Vector Relations and Complex Expressions.* — The vectors representing the sine waves (Fig. 1c) are plotted, in Fig. (1b), for the arbitrary instant $t = 0$. The complex expressions for these vectors are:

$$\left. \begin{aligned} \mathbf{I}_{12} &= \mathbf{I} \epsilon^{j\beta} & (a) \\ \mathbf{E}_{12} &= \mathbf{E} \epsilon^{j(\beta+\phi)} = \mathbf{E}_r + \mathbf{E}_L & (b) \\ \mathbf{E}_r &= R \mathbf{I} \epsilon^{j\beta} = R \mathbf{I}_{12} & (c) \\ \mathbf{E}_L &= X_L \mathbf{I} \epsilon^{j(\beta+\pi/2)} = jX_L \mathbf{I}_{12} & (d) \end{aligned} \right\} \quad (11)$$

Substituting Eqs. (11c) and (11d) in Eq. (11b) we have:

$$\left. \begin{aligned} \mathbf{E}_{12} &= \mathbf{E}_r + \mathbf{E}_L & (a) \\ &= (R + jX_L) \mathbf{I}_{12} = \mathbf{Z}_L \mathbf{I}_{12} & (b) \end{aligned} \right\} \quad (12)$$

where

$$\mathbf{Z}_L = R + jX_L = Z_L \epsilon^{j\phi} \quad (13)$$

Eq. (12a) expresses Kirchhoff's Emf. Law in the vector form. It states that in a series circuit with resistance and inductance the impressed *vector voltage* is equal to the *vector sum* of the drops across the resistance and the inductance. Eq. (13) gives the complex expression for the impedance $\mathbf{Z}_L$. A capital letter is used to denote the numerical value (modulus) of $\mathbf{Z}_L$. A capital letter with a dot under it indicates the complex expression for the impedance. It must be here noted that the impedance $\mathbf{Z}_L$ is not a vector quantity in the sense that $\mathbf{E}$ and $\mathbf{I}$ are vectors. It is an operator which, when multiplied by a vector, changes the latter's magnitude, its physical dimension, as well as its direction. $\mathbf{Z}_L$ shall be hereafter referred to as the *complex impedance* or *impedance operator*.

4. The Voltage and Impedance Triangles. — Eqs. (7) and (9) suggest the existence of two similar right triangles. These triangles shall be termed (a) *the voltage triangle* and (b) *the impedance triangle*.

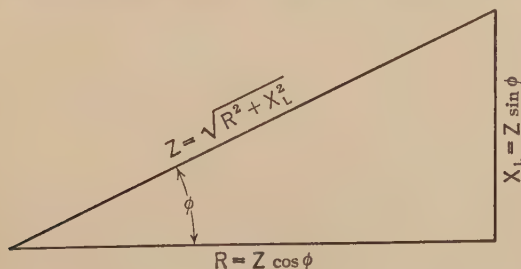


FIG. 2. The impedance triangle of the series circuit Fig. 1a.

(a) *The Voltage Triangle* is shown in Fig. (1b). Its base, altitude, and hypotenuse are $\mathbf{E}_r$, $\mathbf{E}_L$, and $\mathbf{E}_{12}$, respectively; and the angle between $\mathbf{E}_r$ and $\mathbf{E}_{12}$ is ϕ .

The voltage triangle helps to establish in one's mind the fact that $\mathbf{E}_r$ and $\mathbf{E}_L$ are displaced from each other by 90° , that $\mathbf{E}_r$ is in phase with $\mathbf{I}$ and $\mathbf{E}_L$ leads $\mathbf{I}$ by 90° , and that the impressed vector voltage is the hypotenuse of the right triangle or the vector sum of $\mathbf{E}_r$ and $\mathbf{E}_L$.

(b) *The Impedance Triangle* (Fig. 2) has R , X_L , and Z_L for base, altitude, and hypotenuse, respectively. Moreover the angle between Z and R is ϕ .

The concept of the impedance triangle makes clear the relationship expressed in Eqs. (9) and (13).

5. A Series Circuit Consisting of Resistance, Capacitance and a Sine Emf. — (a) *Physical Considerations.* — Fig. 3a represents a series circuit consisting of a resistance R , a capacitance C , and a source of sine

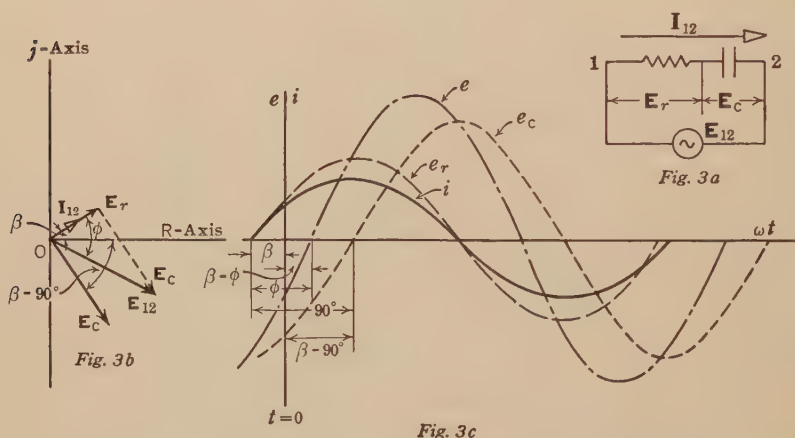


FIG. 3a. A series circuit consisting of R and C and an Emf. $\mathbf{E}_{12}$.

FIG. 3b. Vector diagram for the series circuit Fig. 3a.

FIG. 3c. Sine waves in the series circuit Fig. 3a.

voltage $\mathbf{E}_{12}$. A four-element oscillograph, properly connected in this circuit, will trace waves such as those given in Fig. 3c. An inspection of this figure shows that:

- All the waves are sinusoidal and possess identical frequencies.
- The current wave i leads the wave of impressed voltage e by an angle $\phi < 90^\circ$.
- The voltage drop e_r , across the resistance, is in phase with the current.
- The voltage drop e_c across the capacitance lags the current by 90° .

Should an ammeter and voltmeter be connected in the circuit, and simultaneous readings of voltage and current be taken for different values of E , the following relation would be found:

$$E = Z_c I \quad (14)$$

where Z_c is a coefficient of proportionality between the effective values E and I . Z_c is known as the *impedance* of the circuit, and is expressed in ohms.

The mathematical explanation of these physical facts follows:

(b) *Instantaneous and Effective Value Relations.*—Let the instantaneous value i of the sine current flowing through the circuit, Fig. 3a, be expressed by Eq. (2).

Applying Kirchhoff's Emf. Law we have:

$$e = Ri + \int i \, dt/C \quad (15)$$

Substituting Eq. (2) in (15) and integrating we obtain:

$$\left. \begin{aligned} e &= RI_m \sin(\omega t + \beta) - (I_m/\omega C) \cos(\omega t + \beta) + E_0 \quad (a) \\ &= RI_m \sin(\omega t + \beta) + X_c I_m \sin(\omega t + \beta + 90^\circ) + E_0 \quad (b) \end{aligned} \right\} \quad (16)$$

where E_0 is a constant of integration and represents a continuous Emf. actually connected in series with the sine Emf. This continuous Emf., E_0 , may assume any value whatever between $-\infty$ and $+\infty$.*

Eq. (16b) states that a sine current may be produced in a series circuit with R and C either by a pure sine Emf. or by a sine Emf. in series with a continuous Emf. That this latter case is physically possible may be seen from the simple consideration that a continuous Emf., under established conditions, causes no flow of current through a series circuit comprising capacitance because the capacitor acts as an open circuit. Hence there exists no D.C. component of current, under established conditions, and the only current which flows through the circuit is the sine current. Thus the knowledge that a sine current flows in a series circuit of R and C does not enable one to conclude that the impressed Emf. is sinusoidal. The latter may be a pure sine voltage or it may be a combination of a sine and a continuous Emf. in series. But we know from the oscillograms given in Fig. 3c that the Emf. is sinusoidal and that the drops across the resistance and capacitance are also sinusoidal. We conclude, therefore, that E_0 in this case does not exist and Eq.

* Some authors assert that the term E_0 (Eqs. 16) vanishes after a time. This concept is entirely wrong. If a continuous Emf. E_0 is present in the circuit, there is nothing that can make it vanish except its actual removal by physical means. Time has nothing whatever to do with its vanishing.

(16b) reduces to the form:

$$\left. \begin{aligned} e &= RI_m \sin(\omega t + \beta) + X_c I_m \sin(\omega t + \beta - 90^\circ) & (a) \\ &= E_{mr} \sin(\omega t + \beta) + E_{mc} \sin(\omega t + \beta - 90^\circ) & (b) \end{aligned} \right\} \quad (17)$$

where

$$\left. \begin{aligned} E_{mr} &= RI_m & (a) \\ E_{mc} &= (1/\omega c) I_m = X_c I_m & (b) \end{aligned} \right\} \quad (18)$$

The sum of the two sine waves given in Eq. (17) is a sine wave having an instantaneous value:

$$e = E_m \sin(\omega t + \beta - \phi) \quad (19)$$

where

$$\left. \begin{aligned} E_m &= [E_{mr}^2 + E_{mc}^2]^{\frac{1}{2}} & (a) \\ \phi &= \tan^{-1}(E_{mc}/E_{mr}) & (b) \end{aligned} \right\} \quad (20)$$

The following relation may be obtained by substituting, in Eqs. (20), the values of E_{mr} and E_{mc} from Eqs. (18):

$$\left. \begin{aligned} E_m &= I_m \sqrt{R^2 + X_c^2} = Z_c I_m & (a) \\ \phi &= \tan^{-1}(X_c/R) & (b) \end{aligned} \right\} \quad (21)$$

where

$$Z_c = \sqrt{R^2 + X_c^2} \quad (22)$$

Dividing both sides of Eq. (21a) by $\sqrt{2}$ we obtain:

$$E = Z_c I \quad (23)$$

The foregoing theory offers a complete interpretation of the physical facts quoted in Art. 5a. Thus:

- (a) Eqs. (2), (17) and (19) indicate that all the waves (Fig. 3c) are sine waves of the same frequency.
- (b) A comparison of Eqs. (2) and (19) shows that the current wave leads the voltage wave by an angle ϕ .
- (c) The fact that the voltage drop across the resistance is in phase with the current while that across the reactance lags the current by 90° , is demonstrated by Eqs. (17).

Finally Eq. (23) expresses a relation identical with that given in Eq. (14). Moreover through Eq. (22) we have a means of computing the actual value of the coefficient of proportionality Z_c in terms of the known circuit characteristics R and X_c .

c. *Vector Relations and Complex Expressions.* — Taking an arbitrary instant of time $t = 0$ (Fig. 3c), we obtain the vector diagram (Fig. 3b). The complex expressions for the vectors are:

$$\left. \begin{aligned} \mathbf{I}_{12} &= I \epsilon^{j\beta} & (a) \\ \mathbf{E}_{12} &= E \epsilon^{j(\beta-\phi)} = \mathbf{E}_r + \mathbf{E}_c & (b) \\ \mathbf{E}_r &= RI \epsilon^{j\beta} = R\mathbf{I}_{12} & (c) \\ \mathbf{E}_c &= X_c I \epsilon^{j(\beta-\pi/2)} = -jX_c \mathbf{I}_{12} & (d) \end{aligned} \right\} \quad (24)$$

Substituting Eqs. (24c) and (24d) in Eq. (24b) we have:

$$\begin{aligned} \mathbf{E}_{12} &= \mathbf{E}_r + \mathbf{E}_c & (a) \\ &= (R - jX_c)\mathbf{I}_{12} = Z_c\mathbf{I}_{12} & (b) \end{aligned} \quad (25)$$

where

$$Z_c = R - jX_c = Z_c e^{-j\phi} \quad (26)$$

Eq. (25a) expresses Kirchhoff's Emf. Law in the vector form. In a series circuit with R and C , the impressed *vector voltage* is equal to the *vector sum* of the drops through the resistance and capacitance.

Eq. (26) gives the impedance operator Z_c in complex form.

6. The Voltage and Impedance Triangles. — Eqs. (20), (21), and (22) suggest the existence of two similar triangles, (a) the voltage triangle and (b) the impedance triangle.

(a) *The Voltage Triangle* is shown in Fig. (3b). Its base, altitude, and hypotenuse are respectively $\mathbf{E}_r$, $\mathbf{E}_c$, and $\mathbf{E}_{12}$; and the angle between $\mathbf{E}_r$ and $\mathbf{E}_{12}$ is ϕ . A comparison of Figs. (1b) and (3b) shows that in the former ϕ is an angle of lag but in the latter it is one of lead. These terms, although loose, are often used in electrical engineering literature. Whenever the term lag occurs, it is understood to mean that the current in a circuit lags the impressed voltage. Similarly the term lead is understood to mean that the current leads the impressed voltage.

(b) *The Impedance Triangle* is shown in Fig. (4). It has R , X_c , and Z for its base, altitude, and hypotenuse respectively. Moreover, the

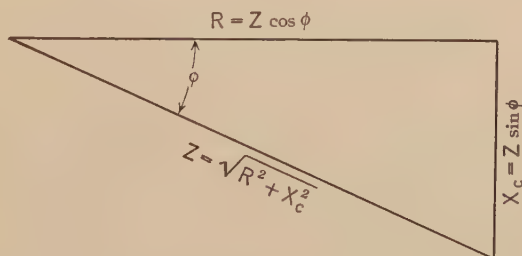


FIG. 4. The impedance triangle of the circuit Fig. 3a.

angle between Z_c and R is ϕ . The graphic picture presented by this triangle is a forcible way of impressing upon the student the relation existing between Z_c , R and X_c .

7. Series Circuit Consisting of Inductance, Capacitance and a Sine Emf. — (a) *Physical Considerations.* — Fig. (5a) represents a source of sine voltage $\mathbf{E}_{12}$ impressed on a circuit consisting of an inductance $\mathcal{L}$ in series with a capacitance C . The sine waves that would be recorded by a four-element oscillograph properly connected in this circuit are drawn in Fig. 5c, from which the following facts may be deduced:

- (a) All the waves are sine waves of the same frequency.
 (b) The voltage drop e_L across the inductance leads the current by 90° .
 (c) The voltage drop e_c across the capacitance lags the current by 90° .
 (d) The impressed voltage e is equal to the difference between e_L and e_c . It is therefore less than one or the other or both of its components.

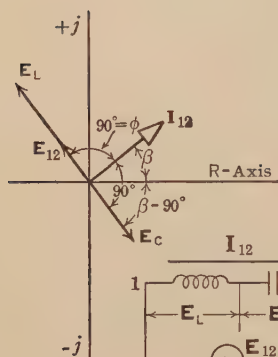


Fig. 5b

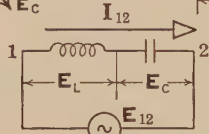


Fig. 5a

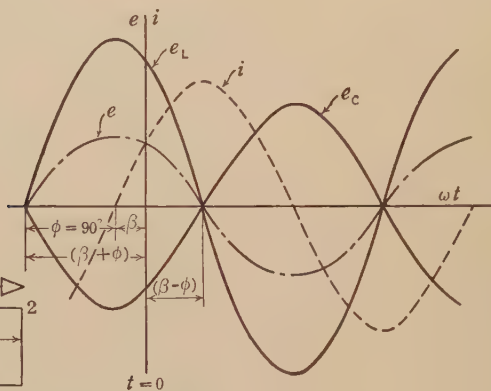


Fig. 5c

FIG. 5a. A series circuit consisting of $\mathcal{L}$ and C and an Emf. E_{12} .

FIG. 5b. Vector diagram for the series circuit Fig. 5a.

FIG. 5c. Sine waves in the series circuit Fig. 5a.

Should an ammeter and a voltmeter be connected in the circuit and simultaneous readings of impressed voltage and current taken for different values of E , the following relation would be found to hold true:

$$E = XI \quad (27)$$

where X is a coefficient of proportionality between E and I and is called the *equivalent reactance* of the circuit. X may be either an inductive or a capacitive reactance depending upon the relative magnitudes of $\mathcal{L}$ and C as well as the frequency of the applied voltage.

(b) *Instantaneous and Effective Value Relations.*—In order to interpret the foregoing physical facts, let the current flowing through the circuit be expressed by Eq. (2). Then by Kirchhoff's Emf. Law:

$$e = \mathcal{L} (di/dt) + \int i dt/C \quad (28)$$

Substituting Eq. (2) in Eq. (28) and performing the required operations we have:

$$e = \mathcal{L}\omega I_m \cos(\omega t + \beta) - (I_m/C\omega) \cos(\omega t + \beta) + E_0 \quad (29)$$

where E_0 , as we have seen, in Art. 5b above, represents a continuous Emf. in series with the impressed sine Emf. The oscillograms, however, show that the impressed Emf. is sinusoidal. We conclude therefore that $E_0 = 0$ and Eq. (28) reduces to:

$$\left. \begin{aligned} e &= (X_L - X_c) I_m \cos(\omega t + \beta) = (X_L - X_c) I_m \sin(\omega t + \beta + 90^\circ) & (a) \\ &= X' I_m \sin(\omega t + 90^\circ) & (b) \\ &= E_m \sin(\omega t + \beta + 90^\circ) & (c) \end{aligned} \right\} \quad (30)$$

$$\text{where} \quad X' = (X_L - X_c) \quad (31)$$

Eq. (30) shows that the impressed voltage e (Fig. 5c) is the numerical difference between the voltage drops across the inductance and capacitance, and that this voltage leads the current by 90° when $X_L > X_c$ and lags when $X_L < X_c$.

(c) *Vector Relations and Complex Expressions.* — For an arbitrary instant of time $t = 0$, the vectors representing the sine waves (Fig. 5c) are given in Fig. 5b. From the vector diagram or from the instantaneous value expressions (Eqs. 2 and 30) the following complex expressions may be written:

$$\left. \begin{aligned} \mathbf{I}_{12} &= I \epsilon^{j\beta} & (a) \\ \mathbf{E}_L &= X_L I \epsilon^{j(\beta + \pi/2)} = jX_L I \epsilon^{j\beta} = jX_L \mathbf{I}_{12} & (b) \\ \mathbf{E}_c &= X_c I \epsilon^{j(\beta - \pi/2)} = -jX_c I \epsilon^{j\beta} = -jX_c \mathbf{I}_{12} & (c) \\ \mathbf{E}_{12} &= \mathbf{E}_L + \mathbf{E}_c = j(X_L - X_c) \mathbf{I}_{12} & (d) \end{aligned} \right\} \quad (32)$$

Eq. (32d) gives Kirchhoff's Emf. Law for a series circuit with $\mathcal{L}$ and C . It states that the impressed *vector voltage* is equal to the *vector sum* of the voltage drops across the inductance and the capacitance (see Fig. 5b). If, in Eq. (32d), $X_L > X_c$ then j is positive, which shows that the equivalent reactance of such a circuit is inductive (see Eq. 23, Chapter VI). On the other hand, if $X_L < X_c$ then j is negative and the reactance is capacitive (see Eq. 32, Chapter VI). The conclusion to be drawn is that the inductive and capacitive reactances of the series circuit (Fig. 5a) may be replaced by one reactance:

$$\text{or} \quad \left. \begin{aligned} X'_L &= X_L - X_c & \text{when } X_L > X_c & (a) \\ X'_c &= X_c - X_L & \text{when } X_c > X_L & (b) \end{aligned} \right\} \quad (33)$$

8. The Phenomenon of Voltage Resonance. — Equating (30a) to (30c) we have:

$$\text{or} \quad \left. \begin{aligned} E_m &= X_L I_m - X_c I_m = \mathbf{E}_{mL} - \mathbf{E}_{mc} & (a) \\ E &= X_L I - X_c I = E_L - E_c & (b) \end{aligned} \right\} \quad (34)$$

Since the effective value of the impressed voltage is by Eq. (34b) equal to the difference between the effective value of the voltage across the inductance and that across the capacitance, it is conceivable that either E_L or E_c , or both, may be greater than E . In other words, if the terminals of a voltmeter are connected respectively across the inductance and capacitance (Fig. 5a), the voltage in either or both cases may be higher than the applied voltage. Such a condition is known as voltage resonance. In dealing with series circuits containing X_L and X_c care should be taken and theoretical computations made in order to ascertain that the voltage components E_L and E_c shall not be excessively high. Such a condition may damage measuring instruments and may prove dangerous to life. With a given impressed voltage of effective value E , the current increases as $X_L - X_c$ approaches zero (see Eq. 34b). Consequently the potential drops $X_L I$ and $X_c I$ also increase. Indeed when $X_L = X_c$, Eq. (34b) reduces to

$$I = E/(X_L - X_c) = (E/0) = \infty \quad (35)$$

Consequently

$$\left. \begin{aligned} E_L &= X_L I = \infty & (a) \\ E_c &= X_c I = \infty & (b) \end{aligned} \right\} \quad (36)$$

Physically this means that when $X_L = X_c$ an infinite current flows through the circuit Fig. 5a irrespective of how small the impressed voltage may be. It further means that the voltage drops across the inductance and capacitance are, under these conditions, both infinitely high. When the characteristics of a series circuit are such that X_L is equal to X_c the circuit is said to be in *perfect resonance*.

Although the phenomenon of voltage resonance is accompanied by an increase in current, this should not be confused with current resonance, which will be treated later. The concept of resonance has nothing to do with the absolute magnitude of the voltage or the current. It is used to describe the apparent anomaly that a voltage component is higher than the total impressed voltage or that a current component is greater than the total current.

It will be interesting to examine the frequency at which perfect resonance occurs in the circuit, Fig. 5a. The condition for resonance is:

$$X_L = X_c \quad (37)$$

Substituting the values of X_L and X_c from Eqs. (20) and (29) Chapter VI, we have:

$$2 \pi f L = 1/(2 \pi f C) \quad (38)$$

Solving Eq. (38) we obtain the value of the resonant frequency:

$$f_r = 1/(2 \pi \sqrt{LC}) \quad (39)$$

9. A Series Circuit Consisting of Resistance, Inductance, Capacitance, and a Sine Emf. — A detailed treatment of a series circuit consisting of R , $\mathcal{L}$, and C is not necessary. It is shown in Art. 8 that the characteristics X_L and X_C may be replaced by an inductive reactance $X'_L = (X_L - X_C)$ when X_L is greater than X_C or by a capacitive reactance $X'_C = (X_C - X_L)$ when X_C is greater than X_L . Hence the circuit shown in Fig. 6a may be replaced either by the circuit Fig. 6b or by the circuit Fig. 6c. These circuits are discussed in Articles 3 and 5 respectively.

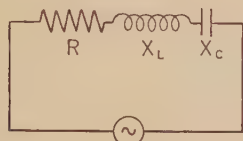


Fig. 6a

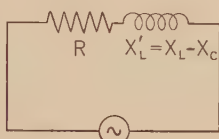


Fig. 6b

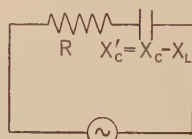


Fig. 6c

Fig. 6a. A series circuit consisting of R , $\mathcal{L}$, C , and sine Emf.

Fig. 6b. A series circuit which is equivalent to the circuit Fig. 6a when $X_L > X_C$.

Fig. 6c. A series circuit which is equivalent to the circuit Fig. 6a when $X_C > X_L$.

There exists, however, a physical difference between the circuit Fig. 6a, and the circuits, Figs. 6b and 6c. This difference lies in the fact that whereas in Figs. 6b and 6c two voltage drops exist, there are three voltage drops in Figs. 6a. Thus applying Kirchhoff's Emf. Law to the circuit Fig. 7a we have:

$$e = Ri + \mathcal{L}(di/dt) + \int i dt/C \quad (40)$$

Let the instantaneous value i of the current flowing through the circuit Fig. 7a be expressed by Eq. (2). Then Eq. (40) becomes:

$$\left. \begin{aligned} e &= RI_m \sin(\omega t + \beta) + \omega \mathcal{L} I_m \cos(\omega t + \beta) \\ &\quad - (I_m/\omega C) \cos(\omega t + \beta) + E_0 \quad (a) \\ &= RI_m \sin(\omega t + \beta) + (X_L - X_C) I_m \cos(\omega t + \beta) + E_0 \quad (b) \\ &= E_m \sin(\omega t + \beta) + E_{mx} \sin(\omega t + \beta + 90^\circ) + E_0 \quad (c) \end{aligned} \right\} \quad (41)$$

Eqs. (41) state that a sine current of the form given by Eq. (2) may be caused in the circuit Fig. 7a by a sine Emf. in a series with a continuous Emf. E_0 which may have any arbitrary value whatever. That such is physically possible may be easily appreciated when it is remembered that a continuous Emf., under steady conditions, causes no flow of current through a series circuit comprising a capacitance because the capacitor acts as an open circuit. Hence although E_0 may be present in the circuit there exists no continuous current under estab-

lished conditions. But by assumption (see Fig. 7a) the impressed Emf. is a sine wave. Hence E_0 , in Eqs. (41), is identically equal to zero, and

$$\left. \begin{aligned} e &= RI_m \sin(\omega t + \beta) + (X_L - X_c)I_m \sin(\omega t + \beta + \pi/2) \quad (a) \\ &= ZI_m \sin(\omega t + \beta \pm \phi) = E_m \sin(\omega t + \beta \pm \phi) \quad (b) \end{aligned} \right\} \quad (42)$$

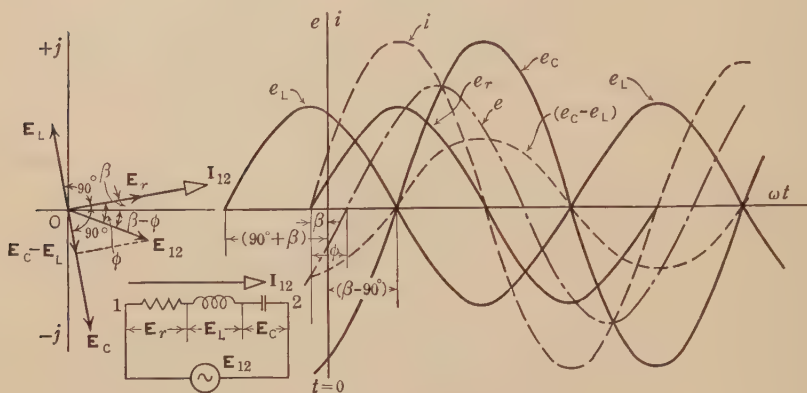


Fig. 7b

Fig. 7a

Fig. 7c

FIG. 7a. A series circuit consisting of R , L , and C and a sine Emf. E_{12} .

FIG. 7b. Vector diagram for the series circuit Fig. 7a.

FIG. 7c. Sine waves in the series circuit Fig. 7a.

Eq. (42b) states that the impressed Emf. in the circuit Fig. 7a has an amplitude $E_m = ZI_m$ and leads or lags the current by an angle ϕ .

In order to determine the values of Z and ϕ in terms of the circuit characteristics let Eq. (42b) be equated to (42a) and let the result be cast into the exponential form. This gives:

$$ZI\epsilon^{j(\beta \pm \phi)} = RI\epsilon^{j\beta} + j(X_L - X_c)I\epsilon^{j\beta} \quad (43)$$

whence:

$$\left. \begin{aligned} Z &= Z\epsilon^{\pm j\phi} = R + j(X_L - X_c) = [\sqrt{R^2 + (X_L - X_c)^2}] \epsilon^{\pm j\phi} \quad (a) \\ \tan \phi &= (X_L - X_c)/R \quad (b) \end{aligned} \right\} \quad (44)$$

It should be here emphasized that Eqs. (42), (43), and (44) are the most general expressions for a series circuit. Thus if $X_c = 0$ these expressions become identical with similar expressions for a series circuit consisting of R and L . If $X_L = 0$ then the equations apply for a circuit with R and C . If $R = 0$ then the equations reduce to those of a circuit with L and C . Finally if any two of the characteristics are equal to zero, the series circuit becomes a simple circuit in which the current and voltage relation may be derived by substituting, in Eqs. (42), (43), and (44), zero for these two characteristics.

10. Series Circuits Consisting of Several Resistances and Reactances.

— Consider a circuit consisting of a vector Emf. $\mathbf{E}_{12}$ and resistances $R_1, R_2, R_3 \dots R_n$ in series with inductive reactances $X_{L1}, X_{L2}, X_{L3} \dots X_{Ln}$ and capacitive reactances $X_{c1}, X_{c2} \dots X_{cn}$. Let a vector current $\mathbf{I}_{12}$ flow through this circuit.

The drops through the various resistances (see Eq. 11c) are:

$$\mathbf{E}_{r1} + \mathbf{E}_{r2} + \dots + \mathbf{E}_{rn} = (R_1 + R_2 + \dots + R_n)\mathbf{I}_{12} = R\mathbf{I}_{12} \quad (45)$$

The drops through the inductive reactances, according to Eq. (11d) are:

$$\mathbf{E}_{L1} + \mathbf{E}_{L2} + \dots + \mathbf{E}_{Ln} = j(X_{L1} + X_{L2} + \dots + X_{Ln})\mathbf{I}_{12} = jX_L\mathbf{I}_{12} \quad (46)$$

The drops through the capacitive reactances (see Eq. 24d) are:

$$\mathbf{E}_{c1} + \mathbf{E}_{c2} + \dots + \mathbf{E}_{cn} = -j(X_{c1} + X_{c2} + \dots + X_{cn}) = -jX_c\mathbf{I}_{12} \quad (47)$$

Now by Kirchhoff's Emf. Law the vector voltage impressed on the circuit is equal to the vector sum of the drops through the characteristics, or:

$$\mathbf{E}_{12} = [R + j(X_L - X_c)]\mathbf{I}_{12} = Z\mathbf{I}_{12} \quad (48)$$

where

$$\left. \begin{aligned} R &= R_1 + R_2 + R_3 + \dots + R_n & (a) \\ X_L &= X_{L1} + X_{L2} + \dots + X_{Ln} & (b) \\ X_c &= X_{c1} + X_{c2} + \dots + X_{cn} & (c) \end{aligned} \right\} \quad (49)$$

We conclude, therefore, that a series circuit comprising several resistances and reactances may be replaced by a circuit consisting of one resistance R whose value is the sum of the resistances, and one inductive reactance X_L having a value equal to the sum of the inductive reactances and one capacitive reactance X_c of magnitude equal to the sum of the values of the capacitive reactances.

In fact such a circuit may be still further simplified by replacing the capacitive and inductive reactances by one inductive reactance $X'_L = X_L - X_c$ if $X_L > X_c$ or by one capacitive reactance $X'_c = X_c - X_L$ if $X_c > X_L$.

Let the values of X_L and X_c in Eqs. (49) be expressed in terms of the angular velocity and the characteristics $\mathcal{L}$ and C respectively (see Eqs. (20) and (29) Chapter VI). We have:

$$\left. \begin{aligned} \omega\mathcal{L} &= \omega\mathcal{L}_1 + \omega\mathcal{L}_2 + \omega\mathcal{L}_3 + \dots + \omega\mathcal{L}_n & (a) \\ (1/\omega C) &= (1/\omega C_1) + (1/\omega C_2) + (1/\omega C_3) + \dots + (1/\omega C_n) & (b) \end{aligned} \right\} \quad (50)$$

whence:

$$\left. \begin{aligned} \mathcal{L} &= \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \dots + \mathcal{L}_n & (a) \\ (1/C) &= (1/C_1) + (1/C_2) + \dots + (1/C_n) & (b) \end{aligned} \right\} \quad (51)$$

Inductances in series may thus be replaced by a single inductance equal to their sum; and capacitances in series may be replaced by a single capacitance whose reciprocal is equal to the sum of the reciprocals of the individual capacitances. Moreover, since Eqs. (51) are independent of ω , they hold true irrespective of the wave form of the impressed voltage.

PROBLEMS*

1. Draw a series circuit consisting of a resistance $R = 5$ ohms, inductance $\mathcal{L} = 0.15$ mh., and capacitance 0.1μ f.; and compute its characteristics R , X_L , and X_C , for the frequencies $f = 50$; $f = 540$; $f = 660$; $f = 3000$ cycles per second. From your results what do you conclude regarding the variation of X_L and X_C with the frequency?

2. Show that Kirchhoff's Emf. Law for a resistance circuit reduces to Ohm's Law.

3a. At 110 volts, a sine current of 10 amperes flows through a series circuit consisting of R and $\mathcal{L}$. Determine the impedance of the circuit.

3b. The resistance of the circuit specified in problem 3a is 8 ohms. What is its inductive reactance?

3c. A series circuit consists of a resistance $R = 6$ ohms and an inductance of 0.05 mh. The circuit is subjected to a 5000-cycle, 10-volt source.

(1) What is the effective value of the current?

(2) Compute the angle of lag ϕ .

(3) Determine the maximum values E_m and I_m .

(4) Write the instantaneous value expressions for e , i , e_r and e_L , assuming that $e = 0$ when $t = 0$.

(5) Draw a vector diagram showing $\mathbf{E}_{12}$, $\mathbf{I}_{12}$, $\mathbf{E}_r$ and $\mathbf{E}_L$.

(6) Write the complex expressions for the vectors specified in part (5).

3d. An oscillogram, taken for the current and voltage of an induction machine, shows that the amplitude of the voltage wave is 150 volts and that of the current wave is 20 amperes. Moreover the current lags the voltage by 25° .

(a) Determine the equivalent resistance, reactance, and impedance of the circuit.

(b) Write the instantaneous value expressions for e and i , taking the zero value of i for $t = 0$.

(c) Give the corresponding complex expressions for $\mathbf{E}_{12}$ and $\mathbf{I}_{12}$.

(d) Draw the sine waves of e and i indicating the instant $t = 0$.

(e) Draw the vector diagram for $\mathbf{E}_{12}$, $\mathbf{E}_L$, $\mathbf{E}_r$ and $\mathbf{I}_{12}$.

3e. An impedance coil has a current passed through it at a uniformly varying rate of 100 amperes per second. When the current is 150 amperes the voltage is 200 volts and when the current is 200 amperes, the voltage is 250 volts. Determine the resistance and inductance of the coil.

3f. A voltage $\mathbf{E}_{12} = 80 + j30$ volts causes a current $\mathbf{I}_{12} = 5 - j2$ amperes to flow in a series circuit. Determine:

(a) The complex impedance of the circuit.

(b) The resistance and reactance of the circuit.

* Throughout this book, unless otherwise specified, the values of voltage and current given are always effective.

3g. A sine voltage $\mathbf{E}_{12} = 70 - j50$ is impressed on a series circuit of impedance $\mathbf{Z}_L = 15 \text{ cis } 30^\circ$ ohms. What is the complex expression for the current?

3h. A current $\mathbf{I}_{12} = 7 + j3$ amperes flows through a series circuit having a resistance $R = 4$ ohms and an inductive reactance $X_L = 8$ ohms. Determine the complex expression for the impressed voltage.

3i. Sketch the sine waves of the voltage and current given in problem 3f and indicate the instant $t = 0$.

3j. A sine voltage $e + je'$, when applied to the terminals of an impedance coil, produces a current $i + ji'$. Determine the resistance and reactance of the coil.

3k. An impedance $\mathbf{Z} = 7 \angle 30^\circ$ is connected across a 50-cycle source of voltage $\mathbf{E}_{12} = 130 + j40$.

- Write the instantaneous value expressions for e and i .
- Give the complex expression for $\mathbf{I}_{12}$.
- Draw the sine waves for e and i indicating the instant $t = 0$.
- Draw the vector diagram showing $\mathbf{I}_{12}$, $\mathbf{E}_{12}$, $\mathbf{E}_r$ and $\mathbf{E}_L$.

3l. In a series circuit R and $\mathcal{L}$ are known. At a certain instant the voltage is e_1 and the current is i and is increasing at the rate of i' amperes per second. At another instant the voltage is e_2 when the current is again i amperes, but is decreasing at the rate of $2 i'$ amperes per second. Find i in terms of e_1 , e_2 and R .*

3m. A sine voltage $\mathbf{E}_{12} = 28 + j12$ volts, when applied to the terminals of an inductance coil, produces a current $\mathbf{I}_{12} = 4 + j1.5$ amperes.

- Express the impedance of the coil in complex form.
- Give the values of R , X_L and Z in ohms.
- Find the effective values E , E_r , E_L and I .
- Draw a vector diagram showing $\mathbf{E}_{12}$, $\mathbf{E}_r$, $\mathbf{E}_L$ and $\mathbf{I}_{12}$.
- Determine the angle ϕ between $\mathbf{E}_{12}$ and $\mathbf{I}_{12}$.
- Compute the maximum values E_m , E_{mr} , E_{mL} and I_m .
- Write the analytical expressions for the instantaneous values e , e_r , e_L and i .

4a. The voltage drops across the resistance and inductive reactance of a series circuit are $E_r = 70$ volts and $E_L = 60$ volts.

- What is the effective value of the voltage impressed on the circuit?
- Determine the angle of lag ϕ between the impressed voltage and the current flowing through the circuit.
- Draw the voltage triangle.

4b. The impedance of a series circuit is 10 ohms and the angle of lag between the voltage and current is 30° .

- Determine the resistance and inductive reactance of the circuit.
- Draw the impedance triangle.

5a. In order to determine the characteristics of a circuit placed inside a closed box an oscillogram was taken at a frequency of 5000 cycles per second. The record revealed that $E_m = 5$ volts. $I_m = 0.7$ amp. Moreover the current leads the voltage by 60° .

- Determine the equivalent R , X , and Z of the circuit and indicate whether X is inductive or capacitive.

* $\mathcal{L}$ and i' do not appear in the answer.

- (b) Write the instantaneous value expressions for e and i , taking the zero value of e for $t = 0$.
- (c) Give the corresponding complex expression for $\mathbf{E}_{12}$ and $\mathbf{I}_{12}$.
- (d) Draw the sine waves of e and i and indicate the instant $t = 0$.
- (e) Draw the vector diagram for $\mathbf{E}_{12}$, $\mathbf{E}_r$, $\mathbf{E}_c$ and $\mathbf{I}_{12}$.

5b. A circuit consisting of $R = 700$ ohms and $C = 0.2\mu$ f. was subjected to a 1000-cycle, 10-volt source. Determine the maximum charge acquired by the condenser in a quarter of a cycle.

5c. A series circuit consists of a resistance $R = 20$ ohms and a capacitance $C = 0.1\mu$ f. A sine voltage of 10 volts at a frequency of 10^5 cycles per second is impressed on this circuit.

- (a) Determine the capacitive reactance and impedance of the circuit.
- (b) Find the effective value of the current.
- (c) Compute the angle of current lead.
- (d) Determine E_m and I_m .
- (e) Compute the voltage drops E_{mr} and E_{mc} .
- (f) Write the instantaneous value expressions for e , i , e_r and e_c , assuming that $t = 0$ when $e = 0$.
- (g) Draw a vector diagram corresponding to part (f).
- (h) Write the complex expressions for all the vectors involved.

5d. A voltage $\mathbf{E}_{12} = 70 - j80$ volts causes a current $\mathbf{I}_{12} = 5 - j2$ amperes to flow in a series circuit.

- (a) Determine the complex impedance of the circuit.
- (b) Give the numerical values of the resistance and reactance of the circuit.
- (c) Is the reactance inductive or capacitive? Why?
- (d) What is the phase angle ϕ between the voltage and current? Is it an angle of lead or of lag?

5e. A sine voltage $\mathbf{E}_{12} = 50 - j70$ volts is impressed on a series circuit of impedance $10 \text{ cis } 30^\circ$ ohms. What is the complex expression for the current?

5f. A current $10 + j5$ amperes flows through a series circuit having a resistance $R = 7$ ohms and a capacitive reactance $X_c = 9$ ohms. Determine the complex expression for the impressed voltage.

5g. Sketch the sine waves of the voltage and current given in problem 5d and indicate the instant $t = 0$.

5h. A sine voltage $\mathbf{E}_{12} = 90 - j50$ volts, when applied to the terminals of a circuit consisting of R and X_c in series, produces a current $\mathbf{I}_{12} = 8 + j12$ amperes.

- (a) Determine the complex impedance of the coil.
- (b) Give the numerical values of R , X_c , and Z_c .
- (c) Compute the effective values E , E_r , E_c , and I .
- (d) Draw a vector diagram showing $\mathbf{E}_{12}$, $\mathbf{E}_r$, $\mathbf{E}_c$, and $\mathbf{I}_{12}$.
- (e) Determine the angle ϕ between $\mathbf{E}_{12}$ and $\mathbf{I}_{12}$.
- (f) Give the maximum values E_m , E_{mr} , E_{mc} , and I_m .
- (g) Write the analytical expressions for the instantaneous values e , e_r , e_c , and i .

6a. 110 volts are impressed on a circuit consisting of a resistance and a capacitance. The voltage drop across the resistance is 50 volts.

- (a) What is the effective value of the drop across the capacitance?
- (b) What is the phase angle between the current and the applied voltage?

6b. The capacitive reactance of a series circuit is 2 ohms and the angle of lead between the current and voltage is 20° . Determine (a) the resistance and (b) the complex impedance of the circuit.

7a. A condenser of 5μ f. is connected in series with an inductance of $\mathcal{L}$ mh. across a 100-volt, 500-cycle circuit. Neglecting the resistance of the circuit, compute the values of $\mathcal{L}$ necessary to produce:

(a) A current of 10 amperes leading.

(b) A current of 5 amperes lagging.

7b. A sine voltage $\mathbf{E} = 100 - j150$ is impressed on a series circuit consisting of an inductive reactance $X_L = 15$ ohms and a capacitive reactance $X_C = 9$ ohms.

(a) Determine the complex expression for the current $\mathbf{I}_{12}$ flowing through the circuit.

(b) Give the complex value of the voltage drops across the inductance and capacitance respectively.

(c) Check the result obtained in part (b) with the equation $\mathbf{E} = \mathbf{E}_L + \mathbf{E}_C$.

(d) Write the instantaneous value expressions for e , i , e_L , and e_C .

(e) Determine the equivalent reactance X'_L .

8a. What value of $\mathcal{L}$ would give perfect resonance in problem 7a?

8b. A circuit consists of an inductive reactance of 20 ohms at 60 cycles in series with a condenser. Assuming that the combination gives perfect resonance, determine the resistance that must be connected in series with X_L and X_C in order that the voltages $\mathbf{E}_L$ and $\mathbf{E}_C$ shall not exceed 500 volts when the combination is subjected to an impressed voltage of 100 volts at 60 cycles.

8c. Determine the resonant frequency of a series circuit consisting of a capacitance $C = 0.05\mu$ f. and an inductance $\mathcal{L} = 0.12$ mh.

9a. A resistance $R = 15$ ohms, inductance $\mathcal{L} = 0.3$ mh., and capacitance $C = 0.2\mu$ f. are connected in series across a 30,000-cycle, 5-volt source.

(a) Compute the inductive and capacitive reactances as well as the impedance of the circuit.

(b) Determine the effective value of the current.

(c) Find the phase angle between the current and voltage.

(d) Compute the amplitudes of the current and voltage.

(e) Write the instantaneous value expressions for the impressed voltage e , the drops e_r , e_L , and e_C , and the current i , assuming that $t = 0$, when e has an instantaneous value of -3 volts, and the slope of the voltage wave is positive.

(f) Draw the corresponding vector diagram showing $\mathbf{E}_{12}$, $\mathbf{I}_{12}$, $\mathbf{E}_r$, $\mathbf{E}_L$, $\mathbf{E}_C$, and the phase angles.

(g) Write the complex expressions for the current, impressed voltage, and the various voltage drops.

9b. Show that when $X_C = 0$, $X_L = 0$, $R = 0$ are substituted successively in Eqs. (40) to (44) these equations become identical with similar expressions derived for series circuits with R and $\mathcal{L}$; R and C ; $\mathcal{L}$ and C , respectively.

9c. Show that when $R = 0$ and $X_L = 0$ are substituted in Eqs. (40) to (44) the resultant expressions become applicable to a simple circuit consisting of a capacitance C .

90 SERIES CIRCUITS WITH CONSTANT CHARACTERISTICS

9d. Show that when $X_L = 0$ and $X_C = 0$ are substituted in Eqs. (40) to (44) these equations express the laws derived for a simple circuit consisting of a resistance R .

9e. Show that when $X_C = 0$ and $R = 0$ are substituted in Eqs. (40) to (44) these equations become applicable to a circuit consisting of a pure inductance $\mathcal{L}$.

10a. A series circuit consists of impedances $Z_1, Z_2, \dots, Z_n$. Show that the equivalent impedance of the circuit is:

$$Z = Z_1 + Z_2 + \dots + Z_n.$$

10b. Two complex impedances $Z_1 = 5 + j7$ and $Z_2 = 8 \text{ cis } 30^\circ$ are connected in series. If a voltage $E_{12} = 200 - j100$ volts is impressed across the combination what will be the complex expression for the current flowing through the circuit?

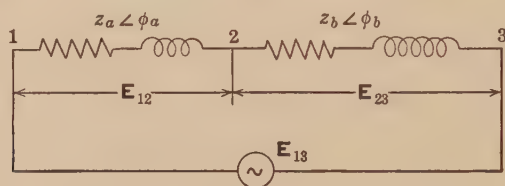


FIG. 8. Two impedances in series.

10c. Fig. 8 represents two impedances connected in series across a sine voltage $E_{13} = E_{13} \epsilon^{j\alpha}$. Show that:

$$E_{12} = (Z_a E_{13} / Z) \epsilon^{j(\alpha - \phi + \phi_a)}$$

where

$$Z = Z_a + Z_b = Z \epsilon^{j\phi}$$

10d. A circuit consists of two impedances $10 \text{ cis } 60^\circ$ and $8 \text{ cis } (-30^\circ)$ connected in series. Determine the equivalent impedance of the combination.

10e. Compute the equivalent reactance at 60,000 cycles per second of a series circuit consisting of two condensers $C_1 = 0.1 \mu \text{ f.}$ and $C_2 = 0.15 \mu \text{ f.}$, and three inductances $\mathcal{L}_1 = 0.1 \text{ mh.}$, $\mathcal{L}_2 = 0.15 \text{ mh.}$, and $\mathcal{L}_3 = 0.25 \text{ mh.}$

10f. Determine the equivalent resistance, inductance, and capacitance of a circuit consisting of: three resistances $R_1 = 3 \text{ ohms}$, $R_2 = 5 \text{ ohms}$, and $R_3 = 7 \text{ ohms}$; inductances $\mathcal{L}_1 = 0.1 \text{ mh.}$, $\mathcal{L}_2 = 0.2 \text{ mh.}$, and $\mathcal{L}_3 = 0.5 \text{ mh.}$; and capacitances $C_1 = 0.05 \mu \text{ f.}$, $C_2 = 0.07 \mu \text{ f.}$, and $C_3 = 0.09 \mu \text{ f.}$

CHAPTER VIII

PARALLEL CIRCUITS WITH CONSTANT CONCENTRATED CHARACTERISTICS

1. Definition. — The characteristics of a circuit are said to be in parallel when the same voltage is across each of them. Fig. 1a represents a parallel circuit because both the resistance and inductance are across the same Emf.

2. Kirchhoff's Current or First Law. — The following is a statement of Kirchhoff's current law for alternating electric circuits:

The sum of the instantaneous values of all the currents flowing toward a point is equal to the sum of the instantaneous values of all the currents flowing away from a point. This law follows from the fact that electricity behaves like an incompressible fluid. Therefore as much current flows toward a point as away from it.

3. A Parallel Circuit with Constant Conductance, Inductance, and a Sine Emf. — (a) *Physical Considerations.* — A parallel circuit (Fig. 1a), consisting of a conductance G and an inductance $\mathcal{L}$, is subjected

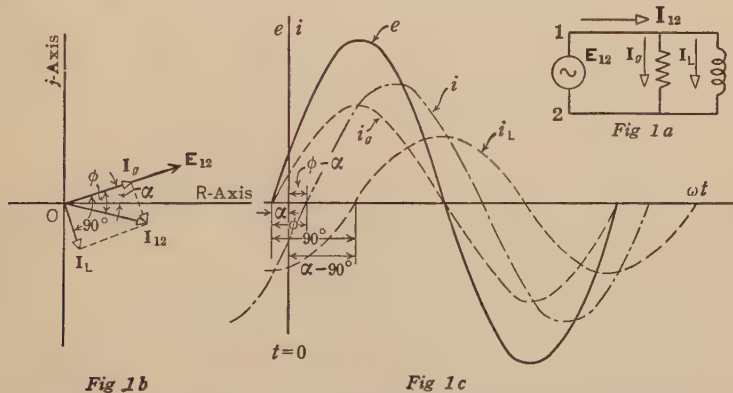


FIG. 1a. A parallel circuit consisting of G , $\mathcal{L}$, and a sine Emf. E_{12} .

FIG. 1b. Vector diagram for the parallel circuit Fig. 1a.

FIG. 1c. Sine waves in the parallel circuit Fig. 1a.

to a sine Emf. E_{12} . The various sine waves that would be obtained by properly connecting a four-element oscillograph in the circuit are shown in Fig. 1c. Observe the following facts:

- (a) The sine current i_g through the conductance is in phase with the applied voltage e .
- (b) The sine current i_L through the inductance lags the applied voltage by 90° .
- (c) The total sine current $i = i_g + i_L$ lags the applied voltage by an angle ϕ which is less than 90° .

If simultaneous readings of the applied voltage and the total current are taken, for various values of the applied voltage, then the following relation would hold true:

$$I = Y_L E \quad (1)$$

where Y_L is a coefficient of proportionality between the effective values E and I , and is called the *admittance* of the parallel circuit. It is expressed in mhos if E is in volts and I in amperes.

(b) *Instantaneous and Effective Value Relations.*—Let the instantaneous voltage impressed on the circuit Fig. 1a be:

$$e = E_m \sin (\omega t + \alpha) \quad (2)$$

This voltage causes a sine current to flow through the conductance G such that (see Art. 5, Chapter VI):

$$i_g = Ge = GE_m \sin (\omega t + \alpha) = I_{mg} \sin (\omega t + \alpha) \quad (3)$$

where

$$I_{mg} = GE_m \quad (4)$$

The same voltage acts on the inductance and, since the current flowing through an inductance lags the impressed voltage by 90° (see Art. 6, Chapter VI), the current i_L is:

$$i_L = B_L E_m \sin (\omega t + \alpha - 90^\circ) = I_{mL} \sin (\omega t + \alpha - 90^\circ) \quad (5)$$

where

$$I_{mL} = B_L E_m \quad (6)$$

Applying Kirchhoff's current law to Fig. 1a we have:

$$\left. \begin{aligned} i &= I_{mg} \sin (\omega t + \alpha) + I_{mL} \sin (\omega t + \alpha - 90^\circ) \quad (a) \\ &= I_m \sin (\omega t + \alpha - \phi) \quad (b) \end{aligned} \right\} \quad (7)$$

where

$$\left. \begin{aligned} I_m &= \sqrt{I_{mg}^2 + I_{mL}^2} \quad (a) \\ \tan \phi &= I_{mL} / I_{mg} \quad (b) \end{aligned} \right\} \quad (8)$$

The result of substituting Eqs. (4) and (6) in (8) is:

$$\left. \begin{aligned} I_m &= E_m \sqrt{G^2 + B_L^2} = Y_L E_m \quad (a) \\ \phi &= \tan^{-1} (B_L / G) \quad (b) \end{aligned} \right\} \quad (9)$$

where

$$Y_L = \sqrt{G^2 + B_L^2} \quad (10)$$

Dividing Eq. (9a) by $\sqrt{2}$ we have:

$$I = Y_L E \quad (1')$$

The reader will perceive that the foregoing treatment offers a complete explanation of the physical facts presented in Art. 3a. Thus Eqs. (3) show that the sine current through the conductance is in phase with the impressed voltage, and Eq. (5) indicates that the current through the inductance lags this voltage by 90° . Again, Eq. (7b) shows that the total current i lags the impressed voltage by $\phi < 90^\circ$. Moreover, Eq. (1') corresponds exactly with Eq. (1). Finally, Eq. (10) gives the actual value of the *admittance* Y_L in terms of the circuit characteristics G and B_L .

(c) *Vector Relations and Complex Expressions.* — The vectors representing the sine waves, for the instant $t = 0$ (Fig. 1c), are plotted in Fig. 1b. The complex expressions for these vectors are:

$$\left. \begin{aligned} \mathbf{E}_{12} &= E\epsilon^{j\alpha} & (a) \\ \mathbf{I}_{12} &= I\epsilon^{j(\alpha-\phi)} = \mathbf{I}_g + \mathbf{I}_L & (b) \\ \mathbf{I}_g &= GE\epsilon^{j\alpha} = G\mathbf{E}_{12} & (c) \\ \mathbf{I}_L &= B_LE\epsilon^{j(\alpha-\pi/2)} = -jB_L\mathbf{E}_{12} & (d) \end{aligned} \right\} \quad (11)$$

Substituting Eqs. (11c) and (11d) in Eq. (11b) we have:

$$\left. \begin{aligned} \mathbf{I}_{12} &= \mathbf{I}_g + \mathbf{I}_L & (a) \\ &= (G - jB_L)\mathbf{E}_{12} = Y_L\mathbf{E}_{12} & (b) \end{aligned} \right\} \quad (12)$$

where

$$Y_L = G - jB_L = Y_L\epsilon^{-j\phi} \quad (13)$$

Eq. (12a) gives Kirchhoff's current law in the vector form. *It states that in a parallel circuit with G , B_L , and $\mathbf{E}_{12}$ the total vector current is equal to the vector sum of the currents flowing in each of the branches.*

Eq. (13) gives the complex expression for the admittance Y_L . A capital letter Y will be used to denote the numerical value of Y_L . A capital letter $\dot{Y}_L$ (with a dot under it) stands for the complex expression given by Eq. (13). Although it is a complex quantity, admittance is not a vector. It is simply a complex operator which, when multiplied by a vector, changes the latter's magnitude, physical dimension, as well as its direction. $\dot{Y}_L$ shall be referred to as the *admittance operator* or as the *complex admittance*.

4. The Current and Admittance Triangles. — Just as in series circuits there exist two similar right triangles for the voltage and impedance, so in parallel circuits there occur two similar right triangles for the

current and admittance. In fact, Eqs. (8), (9) and (10) suggest the existence of these triangles, which are shown in Figs. 1b and 2.

The current triangle (Fig. 1b) has I_g , I_L , and I_{12} for its base, altitude, and hypotenuse, respectively. The angle between I_{12} and I_g is ϕ . The concept of the current triangle gives the student a graphic picture of the fundamental facts that the currents in the conductance and inductance are respectively in phase with and 90° behind the applied voltage, and that the total current lags this voltage by an angle $\phi < 90^\circ$.

The admittance triangle (Fig. 2) has G , B_L , and Y_L for base, altitude, and hypotenuse, respectively. Furthermore the angle between G and Y_L is ϕ . The admittance triangle gives a picture of the numerical relations (see Eq. 10) between G , B_L , and Y_L .

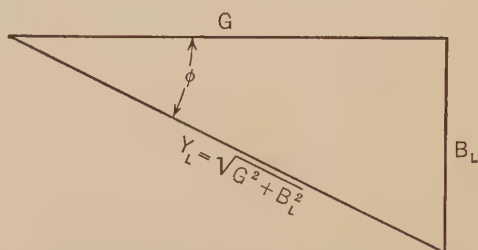


FIG. 2. The admittance triangle of the parallel circuit Fig. 1a.

5. Parallel Circuits with Conductance, Capacitance, and a Sine Emf. — (a) *Physical Considerations.* — Fig. 3a represents a parallel circuit consisting of a conductance G , a capacitance C , and an impressed sine voltage E_{12} . The sine waves, which would be recorded by an oscillograph, are given in Fig. 3c. It will be observed that:

- (a) The sine current i_g through the conductance is in phase with the applied voltage e .
- (b) The sine current i_c through the capacitance leads the applied voltage by 90° .
- (c) The total sine current $i = i_g + i_c$ leads the applied voltage by an angle $\phi < 90^\circ$.

The following relation may be experimentally confirmed by taking current and voltage readings for various values of the impressed voltage:

$$I = Y_c E \quad (14)$$

where Y_c is a coefficient of proportionality between E and I , and is termed the *admittance of the circuit*. Y_c is measured in mhos.

(b) *Instantaneous and Effective Value Relations.* — If the instantaneous value of the voltage impressed on the circuit (Fig. 3c) is ex-

pressed by Eq. (2) then the current flowing through the conductance would be given by Eqs. (3) and that through the capacitance would be expressed by Eqs. (27), Chapter VI. We thus have:

$$\left. \begin{aligned} e &= E_m \sin(\omega t + \alpha) & (a) \\ i_g &= GE_m \sin(\omega t + \alpha) & (b) \\ &= I_{mg} \sin(\omega t + \alpha) \\ i_c &= B_c E_m \sin(\omega t + \alpha + 90^\circ) & (c) \\ &= I_{mc} \sin(\omega t + \alpha + 90^\circ) \end{aligned} \right\} \quad (15)$$

where:

$$\left. \begin{aligned} I_{mg} &= GE_m & (a) \\ I_{mc} &= B_c E_m & (b) \end{aligned} \right\} \quad (16)$$

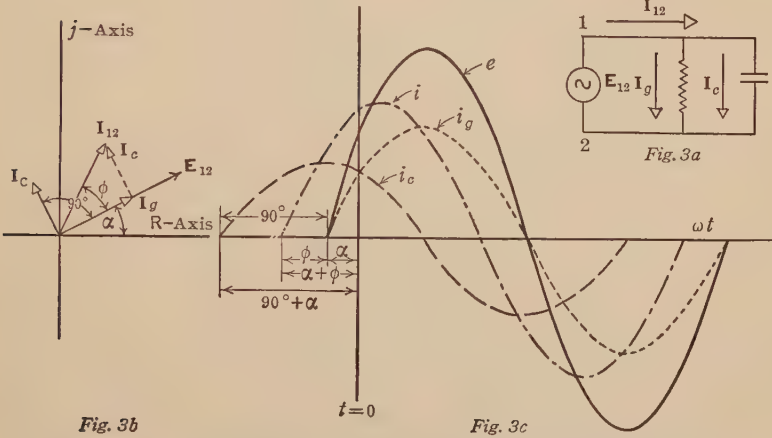


Fig. 3b

Fig. 3c

FIG. 3a. A parallel circuit consisting of G , C , and a sine Emf. E_{12} .

FIG. 3b. Vector diagram for the parallel circuit Fig. 3a.

FIG. 3c. Sine waves in the parallel circuit Fig. 3a.

According to Kirchhoff's current law the instantaneous value, i , of the total current is equal to the sum of the instantaneous values i_g and i_c of the branch currents. Therefore:

$$\left. \begin{aligned} i &= i_g + i_c = I_{mg} \sin(\omega t + \alpha) + I_{mc} \sin(\omega t + \alpha + 90^\circ) & (a) \\ &= I_m \sin(\omega t + \alpha + \phi) & (b) \end{aligned} \right\} \quad (17)$$

where

$$\left. \begin{aligned} I_m &= \sqrt{I_{mg}^2 + I_{mc}^2} & (a) \\ \tan \phi &= I_{mc}/I_{mg} & (b) \end{aligned} \right\} \quad (18)$$

The result of substituting Eqs. (16) in (18) is:

$$\left. \begin{aligned} I_m &= E_m \sqrt{G^2 + B_c^2} = Y_c E_m & (a) \\ \tan \phi &= B_c/G & (b) \end{aligned} \right\} \quad (19)$$

where
$$Y_c = \sqrt{G^2 + B_c^2} \quad (20)$$

Using effective instead of maximum values, Eq. (19a) becomes:

$$I = Y_c E \quad (14')$$

The student will, upon examining the foregoing treatment, convince himself that the results obtained offer a complete explanation of the facts cited in paragraph 5a.

(c) *Vector Relations and Complex Expressions.*—The various vectors representing the sine waves for the arbitrary instant, $t = 0$, are given in Fig. 3b. The complex expressions for these vectors are:

$$\left. \begin{aligned} \mathbf{E}_{12} &= E\epsilon^{j\alpha} & (a) \\ \mathbf{I}_{12} &= YE\epsilon^{j(\alpha+\phi)} = \mathbf{I}_g + \mathbf{I}_c & (b) \\ \mathbf{I}_g &= GE\epsilon^{j\alpha} = G\mathbf{E}_{12} & (c) \\ \mathbf{I}_c &= B_c E\epsilon^{j(\alpha+\pi/2)} = jB_c \mathbf{E}_{12} & (d) \end{aligned} \right\} \quad (21)$$

Substituting Eqs. (21c) and (21d) in Eq. (21b) we have:

$$\left. \begin{aligned} \mathbf{I}_{12} &= \mathbf{I}_g + \mathbf{I}_c & (a) \\ &= (G + jB_c)\mathbf{E}_{12} = Y_c \mathbf{E}_{12} & (b) \end{aligned} \right\} \quad (22)$$

where
$$Y_c = G + jB_c = Y_c \epsilon^{j\phi} \quad (23)$$

Eq. (22a) expresses Kirchhoff's current law in the vector form: namely, the total vector current in a parallel circuit, consisting of G and B_c , is equal to the sum of the vector currents in the separate branches. Eq. (23) gives the complex expression for Y_c .

6. The Current and Admittance Triangles.—Like the parallel circuit discussed in Art. 3, the conductance-capacitance circuit treated in Art. 5 possesses two similar triangles, the current triangle which is shown in Fig. 3b and the admittance triangle represented by Fig. 4.

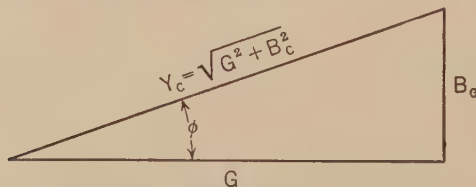


FIG. 4. The admittance triangle of the parallel circuit Fig. 3a.

7. Parallel Circuits with Constant Inductive and Capacitive Susceptances, and a Sine Emf.—(a) *Physical Considerations.*—A parallel circuit (Fig. 5a) consisting of inductive and capacitive susceptances B_L and B_c , is subjected to a sine voltage $\mathbf{E}_{12}$. The sine current

waves and their phase positions, relative to the voltage wave, are shown in Fig. 5c. Observe that:

- (a) The current, i_L , in the inductance lags the applied voltage, e , by 90° .
- (b) The current, i_C , in the capacitance leads the applied voltage, e , by 90° .
- (c) The circuit current, i , is actually the instantaneous algebraic sum of i_L and i_C . Moreover it either leads or lags the applied voltage by 90° depending on whether I_{mC} is greater or less than I_{mL} respectively. In Fig. 5c it has been assumed that $I_{mC} > I_{mL}$. Therefore i leads the applied voltage by 90° .

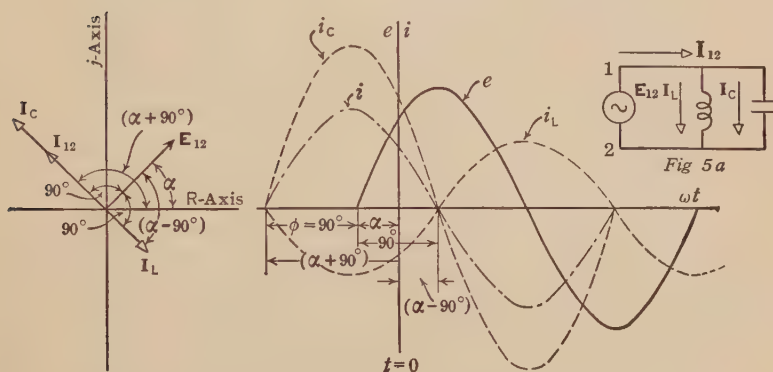


Fig. 5b

Fig. 5c

FIG. 5a. A parallel circuit consisting of $\mathcal{L}$, C , and a sine Emf. E_{12} .

FIG. 5b. Vector diagram for the parallel circuit Fig. 5a.

FIG. 5c. Sine waves in the parallel circuit Fig. 5a.

Should an ammeter and a voltmeter be connected in the circuit and simultaneous reading of impressed voltage and total current taken, for different values of E , the following relation would be found to hold true:

$$I = BE \quad (24)$$

where B is a coefficient of proportionality between E and I and may be termed the *equivalent susceptance* of the circuit. B may be either inductive or capacitive depending upon the relative magnitudes of $\mathcal{L}$ and C as well as the frequency of the Emf. source.

(b) *Instantaneous and Effective Value Relations.*—Let the instantaneous value of the impressed voltage be expressed by Eq. (2). Since this voltage is common to both B_L and B_C , the currents flowing in these branches will have instantaneous values expressed by Eqs. (5) and (15c)

respectively. The total current i , according to Kirchhoff's current law, is:

$$\left. \begin{aligned} i &= B_L E_m \sin(\omega t + \alpha - 90^\circ) + B_c E_m \sin(\omega t + \alpha + 90^\circ) & (a) \\ &= E_m(B_L - B_c) \sin(\omega t + \alpha - 90^\circ) & (b) \\ &= B E_m \sin(\omega t + \alpha - 90^\circ) & (c) \\ &= I_m \sin(\omega t + \alpha - 90^\circ) & (d) \end{aligned} \right\} \quad (25)$$

where

$$\left. \begin{aligned} B &= B_L - B_c & (a) \\ I_m &= I_{mL} - I_{mc} = (B_L - B_c)E_m = B E_m & (b) \\ \text{or } I &= I_L - I_c = (B_L - B_c)E = B E & (c) \end{aligned} \right\} \quad (26)$$

Eq. (25a) represents two sine waves displaced from each other by 180° (see Fig. 5c) and Eq. (25b) shows that the amplitude of the resultant current is actually the difference between the amplitude of the current in $\mathcal{L}$ and that in C . Moreover, the total current i is in phase with i_L if $B_L > B_c$ and in phase with i_c if $B_c > B_L$. Finally Eq. (26a) gives the actual value of the equivalent susceptance B which is the numerical difference between B_L and B_c . We thus have a complete explanation of the physical facts cited above.

(c) *Vector Relations and Complex Expressions.* — The vectors representing the sine waves, (Fig. 5c) are shown in Fig. 5b. The complex expressions for these vectors are:

$$\left. \begin{aligned} \mathbf{E}_{12} &= E \epsilon^{j\alpha} & (a) \\ \mathbf{I}_c &= B_c E \epsilon^{j(\alpha+\pi/2)} = j B_c E \epsilon^{j\alpha} = j B_c \mathbf{E}_{12} & (b) \\ \mathbf{I}_L &= B_L E \epsilon^{j(\alpha-\pi/2)} = -j B_L E \epsilon^{j\alpha} = -j B_L \mathbf{E}_{12} & (c) \\ \mathbf{I}_{12} &= \mathbf{I}_c + \mathbf{I}_L = j(B_c - B_L) \mathbf{E}_{12} & (d) \end{aligned} \right\} \quad (27)$$

Eq. (27d) expresses Kirchhoff's current law for a parallel circuit with $\mathcal{L}$ and C . It states that the total vector current is the vector sum of the currents through the capacitance and inductance.

If, in Eq. (27d), B_c is greater than B_L , then the equivalent susceptance of the circuit is capacitive. On the other hand if $B_L > B_c$ then the equivalent susceptance is inductive. We may therefore conclude that the inductive and capacitive susceptances of the parallel circuit (Fig. 5a) may be replaced by one susceptance:

$$\left. \begin{aligned} B'_c &= B_c - B_L \quad \text{when } B_c > B_L & (a) \\ B'_L &= B_L - B_c \quad \text{when } B_L > B_c & (b) \end{aligned} \right\} \quad (28)$$

This is a significant fact which will be used in the treatment of parallel circuits consisting of G , B_c and B_L .

8. The Phenomenon of Current Resonance. — Since the effective value, I , of the total current (see Eq. 26c) is equal to the difference

between the effective values of the currents through the capacitive and inductive susceptances, it is conceivable that either I_c or I_L or both may be greater than I . Such a condition is known as current resonance. In dealing with circuits containing B_L and B_c in parallel, care should be taken and theoretical computations made in order to ascertain that the current components I_L and I_c shall not be excessively high. This may prove injurious to some parts of the circuit or to the apparatus connected thereto.

Distinction is generally drawn between partial and complete resonance. Resonance is complete when the total current is zero or (see Eq. 26c) when $B_L - B_c = 0$. Resonance is partial when $B_L - B_c \neq 0$.

It would be interesting to state the condition for perfect resonance in terms of $\mathcal{L}, C$, and f . By definition the condition for perfect resonance is:

$$B_L - B_c = 0 \quad \text{or} \quad B_c = B_L \quad (29)$$

Substituting the values of B_c and B_L in terms of $f, \mathcal{L}$, and C we have:

$$1/2 \pi f \mathcal{L} = 2 \pi f C \quad (30)$$

whence

$$f = 1/(2 \pi \sqrt{\mathcal{L}C}) \quad (31)$$

A comparison of Eq. (31) with Eq. (39) Chapter VII, shows that the condition for current resonance in parallel circuits is identical with that for voltage resonance in series circuits.

It should be recalled that the concept of resonance has nothing to do with the absolute magnitude of the current. It is used to describe the apparent anomaly that one or both of the current components are greater in magnitude than the total current supplied by the Emf. source.

9. Parallel Circuits with Constant Conductance, Inductance, and Capacitance. — A detailed treatment for this type of circuits is not necessary. It is shown in Art. 7 that a circuit consisting of $\mathcal{L}$ and C

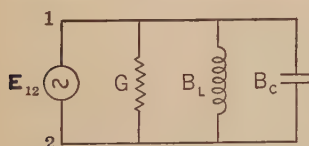


Fig. 6a

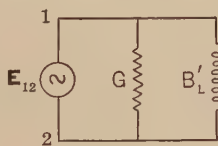


Fig. 6b

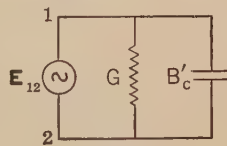


Fig. 6c

FIG. 6a. A parallel circuit consisting of G, B_L, B_c , and a sine Emf. E_{12} .

FIG. 6b. A parallel circuit which is equivalent to the circuit Fig. 6a when $B_L > B_c$.

FIG. 6c. A parallel circuit which is equivalent to the circuit Fig. 6a when $B_c > B_L$.

in parallel may be replaced by one consisting of $B'_L = B_L - B_c$ if $B_L > B_c$ or by one consisting of $B'_c = B_c - B_L$ if $B_c > B_L$. The circuit shown in Fig. 6a may, therefore, be replaced by one or the other

of the circuits shown in Figs. 6b and 6c. These circuits are discussed in Arts. 3 and 5, respectively.

There exists, however, a physical difference between the parallel circuit in Fig. 6a and the parallel circuits in Figs. 6b and 6c. This lies in the fact that in Fig. 6a there are three branch currents whereas in Figs. 6b and 6c only two component currents exist.

The waves, of impressed Emf., total current, and branch currents, are shown in Fig. 7c. The instantaneous value expressions for these waves are:

$$\left. \begin{aligned} e &= E_m \sin(\omega t + \alpha) & (a) \\ i_g &= GE_m \sin(\omega t + \alpha) & (b) \\ i_L &= B_L E_m \sin(\omega t + \alpha - 90^\circ) & (c) \\ i_c &= B_c E_m \sin(\omega t + \alpha + 90^\circ) & (d) \\ i &= i_g + i_L + i_c = I_m \sin(\omega t + \alpha \pm \phi) & (e) \\ &= YE_m \sin(\omega t + \alpha \pm \phi) & (f) \end{aligned} \right\} \quad (32)$$

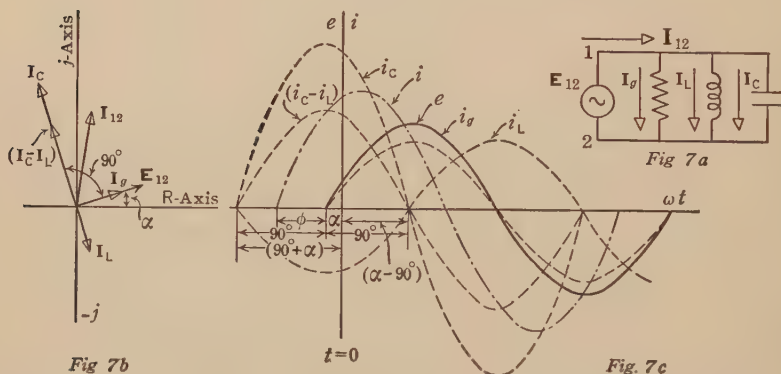


FIG. 7a. A parallel circuit consisting of G , L , and C and a sine Emf. E_{12} .

FIG. 7b. Vector diagram for the circuit Fig. 7a.

FIG. 7c. Sine waves in the circuit Fig. 7a.

The stationary vectors, representing these waves, are given in Fig. 7b and may be analytically expressed as follows:

$$\left. \begin{aligned} E_{12} &= E e^{j\alpha} & (a) \\ I_g &= GE_{12} & (b) \\ I_L &= -jB_L E_{12} & (c) \\ I_c &= jB_c E_{12} & (d) \\ I &= I_g + I_L + I_c & (e) \\ &= [G - j(B_L - B_c)] E_{12} = Y E_{12} & (f) \end{aligned} \right\} \quad (33)$$

where the admittance operator:

$$Y = G - j(B_L - B_c) = Y e^{\pm j\phi} \quad (34)$$

From Eqs. (32e) and (32f) we conclude that the total current flowing through the circuit has an amplitude $I_m = YE_m$ and leads or lags the voltage by an angle $\phi < 90^\circ$.

It should be here emphasized that Eqs. (32), (33) and (34) are the most general expressions for a parallel circuit. Thus if $B_c = 0$ these equations become identical with similar expressions for a parallel circuit with G and B_L . If B_L is zero then the equations apply for a parallel circuit with G and B_c . If $G = 0$ then the equations reduce to those of a circuit with B_L and B_c . Finally, if any two of the characteristics are equal to zero, the parallel circuit becomes a simple circuit in which the current and voltage relationship may be derived by substituting; in Eqs. (32), (33), and (34); zero for these characteristics.

10. Parallel Circuits Consisting of Several Conductances and Susceptances. — Consider a circuit consisting of a vector Emf. $\mathbf{E}_{12}$ and parallel branches having conductances $G_1, G_2 \dots G_n$, inductive susceptances $B_{L1}, B_{L2} \dots B_{Ln}$ and capacitive susceptances $B_{c1}, B_{c2} \dots B_{cn}$. Let the total current through the circuit be $\mathbf{I}_{12}$:

The total current through the conductances is:

$$\mathbf{I}_g = (G_1 + G_2 + G_3 + \dots + G_n)\mathbf{E}_{12} = G\mathbf{E}_{12} \quad (35)$$

The total current through the inductive susceptances is:

$$\mathbf{I}_L = -j(B_{L1} + B_{L2} + \dots + B_{Ln})\mathbf{E}_{12} = B_L\mathbf{E}_{12} \quad (36)$$

and the total current through the capacitive susceptances is:

$$\mathbf{I}_c = j(B_{c1} + B_{c2} + \dots + B_{cn})\mathbf{E}_{12} = B_c\mathbf{E}_{12} \quad (37)$$

By Kirchhoff's current law the vector current $\mathbf{I}_{12}$ is equal to the sum of the vector currents through the various branches or:

$$\mathbf{I}_{12} = \mathbf{I}_g + \mathbf{I}_L + \mathbf{I}_c = \mathbf{E}_{12}[G + j(B_c - B_L)] = Y\mathbf{E}_{12} \quad (38)$$

where

$$\left. \begin{aligned} G &= G_1 + G_2 + G_3 + \dots + G_n & (a) \\ B_L &= B_{L1} + B_{L2} + \dots + B_{Ln} & (b) \\ B_c &= B_{c1} + B_{c2} + \dots + B_{cn} & (c) \end{aligned} \right\} \quad (39)$$

We thus conclude that a parallel circuit comprising several conductances and susceptances may be replaced by a circuit having one conductance equal to the sum of the conductances and one inductive susceptance equal to the sum of the inductive susceptances as well as one capacitive susceptance equal to the sum of the capacitive susceptances. In fact such an equivalent circuit may be still further reduced to one conductance G and one capacitive susceptance $B'_c = B_c - B_L$ if $B_c > B_L$ or one inductive susceptance $B'_L = B_L - B_c$ if $B_L > B_c$.

Next let the values of B_L and B_c in Eqs. (39) be expressed in terms of ω , $\mathcal{L}$, and C . This gives:

$$\left. \begin{aligned} (1/\omega\mathcal{L}) &= (1/\omega\mathcal{L}_1) + (1/\omega\mathcal{L}_2) + \cdots + (1/\omega\mathcal{L}_n) & (a) \\ \omega C &= \omega C_1 + \omega C_2 + \cdots + \omega C_n & (b) \end{aligned} \right\} \quad (40)$$

whence

$$\left. \begin{aligned} (1/\mathcal{L}) &= (1/\mathcal{L}_1) + (1/\mathcal{L}_2) + \cdots + (1/\mathcal{L}_n) & (a) \\ C &= C_1 + C_2 + \cdots + C_n & (b) \end{aligned} \right\} \quad (41)$$

Eqs. (41) show that inductances in parallel may be combined to give an equivalent inductance whose reciprocal is equal to the sum of the reciprocals of the several inductances and that capacitances in parallel may be replaced by a single capacitance equal to their sum.

Since Eqs. (41) are independent of ω , they hold true irrespective of the wave form of the impressed Emf. In fact they hold true for continuous electric circuits.

PROBLEMS

1. Draw a parallel circuit consisting of a resistance $R = 5$ ohms, inductance $\mathcal{L} = 0.2$ mh., and capacitance $C = 0.6\mu$ f.; and compute its characteristics G , B_L , and B_c for the frequencies $f = 25$, and $f = 60$ cycles per second. From your results, what do you conclude regarding the variation of B_L and B_c with frequency?

3a. At 220 volts a sine current of 5 amperes flows through a parallel circuit consisting of a conductance G and an inductive susceptance B_L . Compute the admittance Y of the circuit.

3b. The inductive susceptance of the circuit specified in problem 3a is 0.01 mho. What is its conductance?

3c. The component currents of a parallel circuit are $I_g = 6$ and $I_L = 14$ amperes.

(a) What is the total current flowing through the circuit?

(b) Determine the angle of lag ϕ between the total current and the impressed voltage.

3d. A current $\mathbf{I}_{12} = 8 - j4$ amperes flows through a parallel circuit having a conductance $G = 0.2$ mho and an inductive susceptance $B_L = 0.125$ mho.

(a) What is the complex admittance of the circuit?

(b) Give the complex expression for the impressed voltage.

(c) Write the analytical expressions for the instantaneous values i_g , i_L , i , and e .

(d) Draw the sine waves indicating the instant $t = 0$.

(e) Draw a vector diagram showing amplitudes and proper phase positions of all vectors.

3e. A 60-cycle voltage, $\mathbf{E}_{12} = 40 + j90$, when applied to the terminals of a parallel circuit consisting of a conductance G and inductance $\mathcal{L}$ produces a current $\mathbf{I}_{12} = 8 + j3$ amperes.

(a) Express the admittance Y of the circuit in complex form.

(b) Find the value of $\mathcal{L}$ in millihenries.

(c) Give the numerical value of the admittance Y .

- (d) Determine the effective values E and I .
- (e) Draw a vector diagram showing E_{12} , I_{12} , I_g , and I_L .
- (f) Compute the angle ϕ between the vectors E_{12} and I_{12} .
- (g) Give the maximum values E_m , I_m , I_{mg} , and I_{mL} .
- (h) Write the analytical expressions for the instantaneous values of the sine waves represented by the vectors specified in part (e).

3f. An oscillogram of the impressed voltage and total current flowing through a parallel circuit shows that the voltage and current waves have amplitudes of 150 volts and 15 amperes, respectively. Moreover, the current lags the voltage by 30° .

- (a) Determine the equivalent admittance, conductance, and inductive susceptance of the circuit.
- (b) Write the instantaneous value expressions for e , i_g , i_L , and i , taking the zero-value of e for $t = 0$.
- (c) Give the corresponding complex expressions for the sine waves mentioned in (b).
- (d) Draw the sine waves for e , i_g , i_L , and i , indicating the instant $t = 0$.
- (e) Draw the vector diagram for E_{12} , I_g , I_L , and I_{12} .

3g. A voltage $170 + j35$ produced a current $21 \text{ cis } 35^\circ$. What are the conductance G and susceptance B of the circuit? Is the susceptance inductive or capacitive? Why?

3h. A resistance $R = 8$ ohms and reactance $X_L = 6$ ohms are connected in parallel across a voltage $E = 90 + j30$ volts.

- (a) Express the admittance Y_L in complex form.
- (b) Find the complex expression for the current I_{12} flowing through the load.
- (c) Give the complex expressions for I_g and I_L and check their sum with the answer obtained in part (b).
- (d) Draw a vector diagram showing E_{12} , I_g , I_L , and I_{12} .
- (e) Give the instantaneous value expressions for e , i_g , i_L , and i and draw the sine waves indicating the instant $t = 0$.
- (f) Compute the effective values E , I , I_g and I_L .

4a. The currents through the conductance and inductive susceptance of a parallel circuit have effective values $I_g = 6$ and $I_L = 9$ amperes.

- (a) What is the effective value of the total current flowing through the circuit?
- (b) Determine the angle of lag ϕ between the voltage and current.
- (c) Draw the current triangle.

4b. The admittance of a parallel circuit is 15 mhos and the angle of lag is 60° .

- (a) Determine the conductance and inductive susceptance of the circuit.
- (b) Draw the admittance triangle.

5a. In order to determine the characteristics of a circuit placed inside a closed box an oscillogram was taken at a frequency of 5000 cycles per second. The record revealed that $E_m = 5$ volts, $I_m = 0.7$ amp. and a current lead of 60° between e and i .

- (a) Determine the equivalent Y , G , and B of the circuit, indicating whether B is inductive or capacitive.
- (b) Write the instantaneous value expressions for e , i , i_g , and i_c , taking the zero value of e for $t = 0$.
- (c) Draw the sine waves specified in part (b).

104 PARALLEL CIRCUITS WITH CONSTANT CHARACTERISTICS

- (d) Give the corresponding complex expressions for $\mathbf{E}_{12}$, $\mathbf{I}_{12}$, $\mathbf{I}_g$ and $\mathbf{I}_c$.
- (e) Draw a complete vector diagram.

5b. A parallel circuit consists of a conductance $G = 0.2$ mho and a capacitance $C = 0.1 \mu$ f. A sine voltage having an amplitude $E_m = 2$ volts and a frequency $f = 5 \times 10^6$ cycles per second is impressed on the circuit.

- (a) Compute the complex admittance of the circuit.
- (b) Find the effective value of the current.
- (c) Compute the angle of current lead.
- (d) Determine I_{mg} and I_{mc} .
- (e) Write the instantaneous value expressions for e , i , i_g , and i_c , assuming that $t = 0$ when $i = 0$.
- (f) Draw a vector diagram corresponding to part (e).
- (g) Write the complex expressions for all the vectors appearing in the vector diagram specified in part (f).

5c. A sine voltage $\mathbf{E}_{12} = 40 - j90$ volts is impressed on a parallel circuit having an admittance $\mathbf{Y} = 0.15 \text{ cis } 60^\circ$. What is the complex expression for the current?

5d. A current $6 + j15$ amperes flows through a parallel circuit having a conductance $G = 0.3$ mho and a capacitive susceptance $B_c = 0.1$ mho. Give the complex expression for the impressed voltage.

5e. A sine voltage $\mathbf{E}_{12} = 70 - j80$ volts when applied to the terminals of a circuit produces a current $\mathbf{I}_{12} = 12 - j5$ amperes.

- (a) Determine the complex admittance of the circuit.
- (b) Give the numerical values of G , B_c and Y_c .
- (c) Determine the effective values E , I_g , I_c , and I .
- (d) Draw a vector diagram showing $\mathbf{E}_{12}$, $\mathbf{I}_g$, $\mathbf{I}_c$, and $\mathbf{I}_{12}$.
- (e) Compute the angle ϕ between $\mathbf{E}_{12}$ and $\mathbf{I}_{12}$.
- (f) Determine the values E_m , I_m , I_{mg} , and I_{mc} .
- (g) Write the analytical expressions for the instantaneous values, e , i , i_g , and i_c .

6. Indicate the current triangles and draw the admittance triangles for problems 5a to 5e.

7a. A condenser of 5μ f. is connected in parallel with an inductance of $\mathcal{L}$ mh. across a 100-volt, 500-cycle source. Neglecting the resistance of the connecting lead and the induction coil, compute the values of $\mathcal{L}$ necessary to produce a circuit current of 5 amperes lagging.

7b. A 10,000-cycle sine voltage $\mathbf{E} = 10 \angle -60^\circ$ is impressed on a parallel circuit consisting of an inductance $\mathcal{L} = 0.2$ mh. and a capacitance $C = 0.5 \mu$ f.

- (a) Compute the equivalent susceptance B of the circuit. Is B inductive or capacitive? Why?
- (b) Determine the complex expressions for $\mathbf{I}_c$, $\mathbf{I}_L$, and $\mathbf{I}_{12}$.
- (c) Check the value of $\mathbf{I}_{12}$ by adding $\mathbf{I}_c$ and $\mathbf{I}_L$.
- (d) Write the instantaneous value expressions for e , i , i_L , and i_c .

8a. What value of $\mathcal{L}$ in problem 7a would give perfect current resonance?

8b. Determine the resonant frequency of the circuit specified in problem 7b.

9a. A resistance $R = 20$ ohms, inductance $\mathcal{L} = 0.3$ mh. and capacitance $C = 0.2 \mu$ f. are connected in parallel across a 10,000-cycle, 5-volt source.

- (a) Compute the conductance, inductive and capacitive susceptances, as well as the admittance of the circuit.
- (b) Determine the effective values I_g , I_L , I_c , and I_{12} and the amplitudes I_{mg} , I_{mL} , I_{mc} , and I_m .
- (c) Find the phase angle ϕ between $\mathbf{E}_{12}$ and $\mathbf{I}_{12}$.
- (d) Write the instantaneous value expressions for the impressed voltage e and the currents i_g , i_L , i_c , i , and e , assuming that $t = 0$ when e has an instantaneous value $= -3$ volts, and the slope of the curve is negative.
- (e) Draw the corresponding diagram showing all vectors and phase angles.
- (f) Write the complex expressions for the impressed voltage and all the vector currents.

9b. Show that when $B_c = 0$, $B_L = 0$, and $G = 0$ are substituted successively in Eqs. (32), (33), and (34) these equations become identical with similar expressions derived for parallel circuits with G and B_L ; G and B_c ; B_L and B_c respectively.

9c. Show that when $G = 0$ and $B_L = 0$ are substituted in Eqs. (32), (33), and (34), the resultant expressions become applicable to a simple circuit consisting of a capacitance C .

9d. Show that when $B_L = 0$ and $B_c = 0$ are substituted in Eqs. (32), (33), and (34), these equations reduce to those of a simple circuit consisting of a conductance G .

9e. Show that the Emf.-current relationship, in a simple circuit consisting of an inductance $\mathcal{L}$, may be derived by substituting, in Eqs. (32), (33), and (34), $B_c = 0$ and $G = 0$.

10a. A parallel circuit consists of admittances $Y_1, Y_2, Y_3, \dots, Y_n$. Show that the equivalent admittance of the circuit is:

$$Y = Y_1 + Y_2 + \dots + Y_n$$

10b. Three admittances $12 - j7$, $6 \angle (-20^\circ)$, and $9 \text{ cis } (-60^\circ)$ are connected in parallel.

- (a) Determine the equivalent admittance of the combination.
- (b) Draw the admittance triangle.

10c. Three admittances $0.2 + j0.7$, $0.3 \angle 60^\circ$, and $0.5 \text{ cis } 30^\circ$ are connected in parallel. What is the equivalent admittance?

10d. Give the equivalent $\mathcal{L}$ and C of a parallel circuit consisting of inductances $\mathcal{L}_1 = 0.005 \text{ mh.}$, $\mathcal{L}_2 = 0.007 \text{ mh.}$, and $\mathcal{L}_3 = 0.002 \text{ mh.}$, and capacitances $C_1 = 1 \mu \text{ f.}$, $C_2 = 2 \mu \text{ f.}$ and $C_3 = 3 \mu \text{ f.}$

10e. The following circuit characteristics are all connected in parallel:

$R_1 = 0.2 \text{ ohm}$	$\mathcal{L}_1 = 0.001 \text{ mh.}$	$C_1 = 5 \mu \text{ f.}$
$R_2 = 0.4 \text{ ohm}$	$\mathcal{L}_2 = 0.003 \text{ mh.}$	$C_2 = 6 \mu \text{ f.}$
$R_3 = 0.6 \text{ ohm}$	$\mathcal{L}_3 = 0.005 \text{ mh.}$	$C_3 = 7 \mu \text{ f.}$

Give the constants G and B of the equivalent circuit if the frequency of the impressed voltage is 10^5 cycles per second.

CHAPTER IX

SERIES OR PARALLEL CIRCUITS WITH VARIABLE CONCENTRATED CHARACTERISTICS

1. Introduction. — It is assumed in Chapters VII and VIII that the characteristics (R and X or G and B) of a circuit remain constant. Such is not always the case. Problems arise in practice where the characteristics vary within a certain range between zero and infinity. Such a variation may be due (1) to the actual addition or elimination from the circuit of R , $\mathcal{L}$, and C , or (2) to a change in the frequency of the Emf. source, or (3) to both causes. An example of both types of change may be had in the operation of a phase-wound induction motor. Here the resistance connected in series with the rotor is gradually diminished as the rotor approaches its rated speed. Simultaneously the frequency of the induced Emf. in the rotor decreases thereby causing a change in the value of the rotor reactance.

With a constant impressed sine voltage any variation in the circuit characteristics causes a corresponding variation not only in the magnitude but also in the phase position of the current. On the other hand, if a constant sine current be maintained, any change in the value of the characteristics would necessitate a corresponding change in the magnitude and phase position of the impressed voltage. Two problems, therefore, arise:

- (a) To find the locus of the terminus of the *variable current vector* in a circuit with a *constant impressed vector voltage* when one or more of the circuit characteristics vary.
- (b) To find the locus of the terminus of the *variable voltage vector* in a circuit with a *constant vector current* when one or more of the circuit characteristics are varied.

Two advantages accrue from knowing the laws of variation of the current or voltage as the circuit characteristics vary:

- (a) Results may be predicted as in the case of the circle diagram of an induction motor.
- (b) The necessity of making numerical computations, in order to ascertain the variable vector, is obviated.

Throughout this treatment the transient term, which naturally accompanies any change in the circuit characteristics, will be neglected.

We shall simply concern ourselves with the steady state which is established after the change has occurred.

The various circuits which might be considered are:

(a) Series circuits consisting of

- (1) r and x_L
- (2) r and x_c
- (3) x_L and x_c
- (4) r , x_L , and x_c

(b) Parallel circuits consisting of

- (1) g and b_L
- (2) g and b_c
- (3) b_L and b_c
- (4) g , b_L , and b_c

On account of the similarity in the solution of these circuits only the first is treated in the text. The remaining cases are given as exercises to the student.

2. A Series Circuit Consisting of a Sine Emf., a Variable r or (and)

$\mathbf{x}_L$. — The following problems arise:

(a) To find the locus of the terminus of the *variable current vector* in a circuit with *constant impressed vector voltage* when:

- (1) R is constant and x_L is variable.
- (2) r is variable and X_L is constant.
- (3) both r and x_L are variable.

(b) To find the locus of the terminus of the *variable voltage vector* in a circuit with *constant vector current* when:

- (1) R is constant and x_L is variable.
- (2) r is variable and X_L is constant.
- (3) both r and x_L are variable.

CASE a_1 : $\mathbf{E}_{12}$ AND R CONSTANT; $\mathbf{I}_{12}$ AND x_L VARIABLE. — Fig. 1a shows a series circuit with a constant resistance R , a variable inductive reactance x_L and a constant voltage vector $\mathbf{E}_{12}$. As the inductive reactance varies, the current through the circuit varies both in magnitude and phase angle. It is required to find the locus of the terminus of the vector $\mathbf{I}_{12}$.

Solution. — Let variable and constant quantities be represented by small and capital letters respectively, and let all vectors be expressed in the orthogonal form. Thus:

$$\left. \begin{aligned} \mathbf{I}_{12} &= i_1 + j\dot{i}_2 & (a) \\ z &= R + jx_L & (b) \\ \mathbf{E}_{12} &= E_1 + jE_2 & (c) \end{aligned} \right\} \quad (1)$$

Eq. (12b), Chapter VII, gives the relation between the current and voltage in such a circuit:

$$(i_1 + ji_2)(R + jx_L) = E_1 + jE_2 \quad (2)$$

The most evident means of determining the locus of the terminus of I_{12} is to assume various values of x_L and compute corresponding values of $i_1 + ji_2$ by the use of Eq. (2). If this is done, the termini of the various vectors I_{12} would lie on the semicircle, Fig. 1b. We conclude, therefore, that this semicircle is the locus.

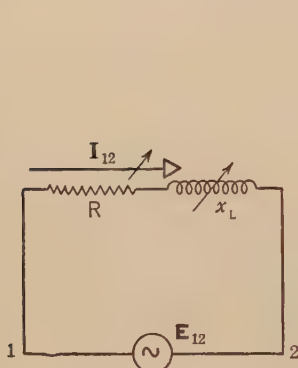


Fig. 1a

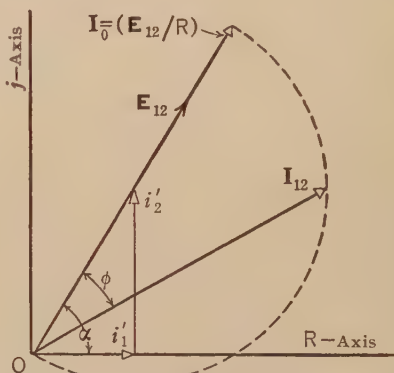


Fig. 1b

FIG. 1a. A series circuit in which E_{12} and R are constant, x_L and I_{12} are variable.

FIG. 1b. Locus of the terminus of the variable current vector I_{12} .

Although this method yields the desired result it is rather crude and involves waste of time. The following is a more direct solution of the problem.

Expand Eq. (2) and equate reals and imaginaries:

$$\left. \begin{aligned} Ri_1 - x_L i_2 &= E_1 & (a) \\ Ri_2 + x_L i_1 &= E_2 & (b) \end{aligned} \right\} \quad (3)$$

Multiply Eq. (3a) by i_1 and (3b) by i_2 and add:

$$R(i_1^2 + i_2^2) - E_1 i_1 - E_2 i_2 = 0^* \quad (4)$$

* The student may obtain Eq. (4) in a form which is familiar to him from his study of analytic geometry by replacing the axes of reals and imaginaries by the coordinate axes x and y respectively. Since i_1 is a variable along the x -axis it may be replaced by x , and since i_2 is a variable along the y -axis it may be replaced by y . Then Eq. (4) assumes the form $Rx^2 + Ry^2 - E_1 x - E_2 y = 0$, which is a circle having a diameter given by Eq. (5) and a center at the point $x' = E_1/2R$ and $y' = E_2/2R$.

Remembering that i_1 and i_2 are variables along the axes of reals and imaginaries respectively, we infer that Eq. (4) represents a circle (Fig. 1b) with diameter:

$$I_0 = \sqrt{(E_1^2 + E_2^2)}/R = E/R \quad (5)$$

The center of the circle is at the point $i'_1 = (E_1/2 R)$ and $i'_2 = (E_2/2 R)$. Moreover, since $i'_1/i'_2 = E_1/E_2$, the center of the circle lies along the voltage vector $\mathbf{E}_{12}$.

We thus conclude that in the circuit, Fig. 1a, the current vector $\mathbf{I}_{12}$ varies in such a manner that its terminus always lies on the circle, Fig. 1b.

Physically the current cannot lead the voltage because the circuit is inductive. Hence the semicircle, Fig. 1b, is the locus.

Let us examine this locus for the limiting values $x_L = 0$, $x_L = \infty$, and for the intermediate values $0 < x_L < \infty$.

(1) If, in Fig. 1a, $x_L = 0$, then the circuit consists of a pure resistance. Consequently the current $I_0 = E/R$ and is in phase with the voltage $\mathbf{E}_{12}$. This is the diameter of the circle, Fig. 1b.

(2) When $x_L = \infty$ the current is zero because the impedance of the circuit is infinite. Thus the current vector dwindles to the point O , Fig. 1b.

(3) As x_L assumes values between zero and infinity the magnitude of the current vector becomes less than I_0 because the impedance of the circuit increases. Moreover the angle of lag ϕ assumes values between zero and 90° . (See the vector $\mathbf{I}_{12}$, Fig. 1b.)

CASE a_2 : $\mathbf{E}_{12}$ AND X_L CONSTANT; $\mathbf{I}_{12}$ AND r VARIABLE. — The relation between the voltage and current in Fig. 2a is expressed as follows:

$$(i_1 + ji_2)(r + jX_L) = E_1 + jE_2 \quad (6)$$

Expanding Eq. (6) and equating reals and imaginaries we obtain:

$$\left. \begin{aligned} ri_1 - X_L i_2 &= E_1 & (a) \\ ri_2 + X_L i_1 &= E_2 & (b) \end{aligned} \right\} \quad (7)$$

The result of multiplying Eq. (7a) by i_2 and (7b) by i_1 and subtracting is:

$$X_L(i_1^2 + i_2^2) - i_1 E_2 + i_2 E_1 = 0 \quad (8)$$

which is the equation of a circle (Fig. 2b) having a diameter:

$$I_0 = [\sqrt{(E_1^2 + E_2^2)}/X_L] = E/X_L \quad (9)$$

The center of the circle is at the point $i'_1 = E_2/2 X_L$ and $i'_2 = -E_1/2 X_L$. Moreover, since $(i'_1/i'_2) = -(E_2/E_1)$, the angles α and β (see Fig. 2b) are complementary. Hence the diameter of the circle makes an angle of 90° with the voltage vector $\mathbf{E}_{12}$.

Physically the current vector cannot lag E_{12} by more than 90° . Therefore the semicircle Fig. 2b is the locus of the terminus of the variable current vector.

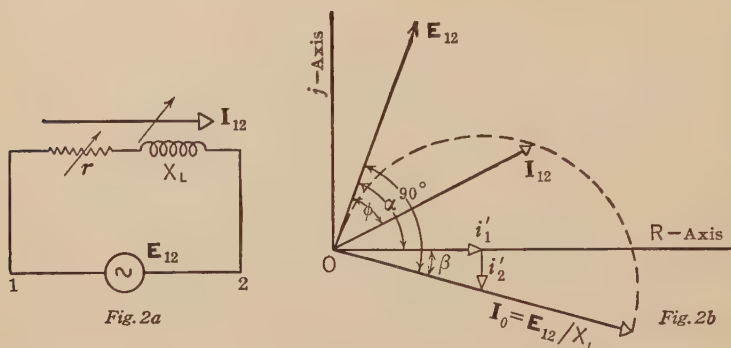


Fig. 2a. A series circuit in which E_{12} and X_L are constant, r and I_{12} are variable.

Fig. 2b. Locus of the terminus of the variable current vector I_{12} .

We shall examine this locus for the limiting values $r = 0$, $r = \infty$, and for the intermediate values $0 < r < \infty$.

(1) When $r = 0$ the circuit (Fig. 2a) consists of only the inductance X_L . Hence the current has a modulus $I_0 = E/X_L$ and lags the voltage by 90° . This is the diameter of the circle, Fig. 2b.

(2) When $r = \infty$ the current is zero. This is represented by the point 0 on the circle.

(3) As r assumes values between zero and infinity the modulus of the current vector ranges between I_0 and 0 and the phase angle ϕ becomes less than 90° as shown by the vector I_{12} , Fig. 2b.

CASE a_3 : E_{12} CONSTANT, r , x_L , AND I_{12} VARIABLE. — When both r and x_L vary the locus becomes indefinite. Indeed the locus of the terminus of I_{12} depends altogether upon the manner in which r and x_L vary simultaneously. As a simple illustration consider the case (Fig. 3a) where r and x_L so vary that:

$$r = ax_L + R_0 \quad (10)$$

where a and R_0 are constants.

The relation between the constant voltage and the variable current is:

$$(i_1 + ji_2)(r + jx_L) = E_1 + jE_2 \quad (11)$$

Expanding Eq. (11) and equating reals and imaginaries we have:

$$\left. \begin{aligned} ri_1 - x_L i_2 &= E_1 & (a) \\ ri_2 + x_L i_1 &= E_2 & (b) \end{aligned} \right\} \quad (12)$$

The result of multiplying Eq. (12a) by i_1 and (12b) by i_2 and adding is:

$$r = (i_1 E_1 + i_2 E_2) / (i_1^2 + i_2^2) \quad (13a)$$

Multiplying Eq. (12a) by i_2 and (12b) by i_1 and subtracting we have

$$x_L = (i_1 E_2 - i_2 E_1) / (i_1^2 + i_2^2) \quad (13b)$$

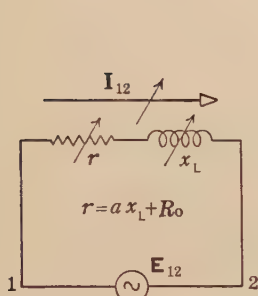


Fig. 3a

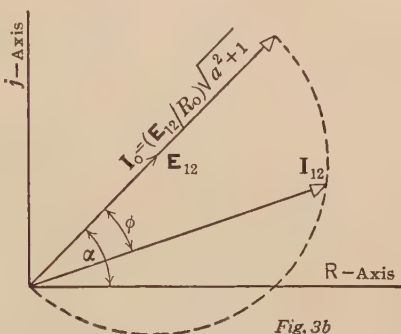


Fig. 3b

FIG. 3a. A series circuit in which $\mathbf{E}_{12}$ is constant, r , x_L , and $\mathbf{I}_{12}$ are variable.

FIG. 3b. Locus of the terminus of the variable current vector $\mathbf{I}_{12}$ when r and x_L so vary that $r = ax_L + R_0$.

Substituting Eq. (10) in (13a) and solving for x_L we have:

$$x_L = [(i_1 E_1 + i_2 E_2) - R_0(i_1^2 + i_2^2)] / a(i_1^2 + i_2^2) \quad (14)$$

Equating (13b) and (14) and simplifying we have:

$$i_1^2 + i_2^2 + [(aE_2 - E_1)/R_0]i_1 - [(aE_1 + E_2)/R_0]i_2 = 0 \quad (15)$$

which is the equation of a circle* (Fig. 3b) whose center is at the point $i'_1 = (E_1 - aE_2)/2R_0$ and $i'_2 = (aE_1 + E_2)/2R_0$ and whose diameter I_0 is such that:

$$I_0 = (E/R_0) \sqrt{(a^2 + 1)} \quad (16)^*$$

CASE b_1 : $\mathbf{I}_{12}$ AND R CONSTANT; $\mathbf{E}_{12}$ AND x_L VARIABLE. — The equation expressing the relationship between the voltage and current in Fig. 4a is:

$$(R + jx_L)(I_1 + jI_2) = e_1 + je_2 \quad (17)$$

* As a check on the accuracy of this treatment consider the condition where $a = 0$. This means (see Eq. 10) that $R = R_0 = a$ constant. In other words this case is reduced to case a_1 . Eq. (15) becomes:

$$i_1^2 + i_2^2 - (E_1/R_0)i_1 - (E_2/R_0)i_2 = 0 \quad (15')$$

which is the equation of a circle with center at $i'_1 = E_1/2R_0$ and $i'_2 = E_2/2R_0$ and diameter $I_0 = E/R_0$, that is, the identical circle found in case a_1 .

Expanding Eq. (17) and equating reals and imaginaries we have:

$$\left. \begin{aligned} RI_1 - x_L I_2 &= e_1 & (a) \\ RI_2 + x_L I_1 &= e_2 & (b) \end{aligned} \right\} \quad (18)$$

Multiplying Eq. (18a) by I_1 and Eq. (18b) by I_2 and adding we obtain:

$$I_1 e_1 + I_2 e_2 - R(I_1^2 + I_2^2) = 0 \quad (19)$$

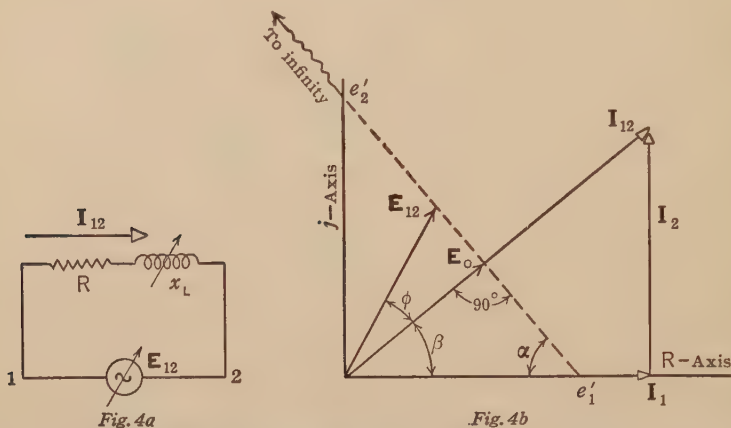


FIG. 4a. A series circuit in which R and I_{12} are constant, x_L and E_{12} are variable.

FIG. 4b. Locus of the terminus of the variable voltage vector E_{12} .

This is the equation of a straight line* (Fig. 4b) intersecting the R - and j -axes respectively at the points:

$$\left. \begin{aligned} e'_1 &= RI/I_1 & (a) \\ e'_2 &= RI/I_2 & (b) \end{aligned} \right\} \quad (20)$$

Where

$$I = \sqrt{I_1^2 + I_2^2} \quad (21)$$

Dividing Eq. (20a) by (20b) we have:

$$(e'_1/e'_2) = (I_2/I_1) \quad (22)$$

Since the intercepts e'_1 and e'_2 are inversely proportional to the projections of I_{12} along the coördinate axes, the angles α and β (Fig. 4b) are complementary and therefore the locus is perpendicular to the current vector.

Physically the voltage E_{12} cannot lag the current because the circuit is inductive. Hence the locus starts at the terminus of E_0 and extends to infinity as shown in Fig. 4b.

* The reader may obtain Eq. (19) in a form which is probably more familiar to him by replacing the variable e_1 and e_2 by x and y as was done in case a_1 . Then Eq. (19) assumes the form $ax + by + c = 0$ which is the equation of a straight line.

We shall examine this locus for the cases $x_L = 0$; $x_L = \infty$ and $0 < x_L < \infty$.

When x_L (Fig. 4a) is zero, the impedance of the circuit is a minimum consequently the voltage required to supply the constant current $\mathbf{I}_{12}$ is a minimum and is in phase with $\mathbf{I}_{12}$. This is the vector voltage $\mathbf{E}_0$, Fig. 4b. As x_L increases both the impressed voltage and the phase angle ϕ increase. This is shown by the position and magnitude of $\mathbf{E}_{12}$, Fig. 4b. As x_L approaches infinity the vector $\mathbf{E}_{12}$ becomes infinitely large and the phase angle ϕ approaches 90° .

CASE b_2 : $\mathbf{I}_{12}$ AND x_L CONSTANT; $\mathbf{E}_{12}$ AND r VARIABLE. — The relationship between the voltage and current (Fig. 5a) is:

$$(r + jX_L)(I_1 + jI_2) = e_1 + je_2 \quad (23)$$

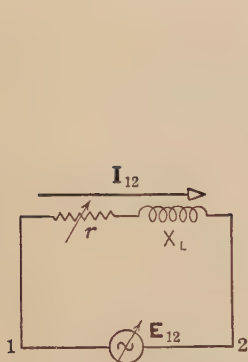


Fig. 5a

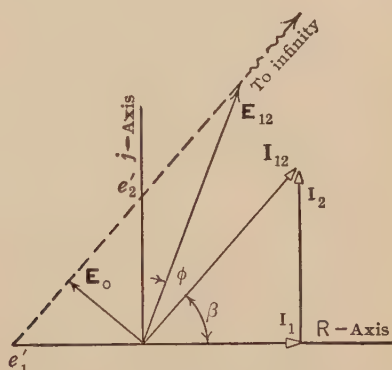


Fig. 5b

FIG. 5a. A series circuit in which X_L and $\mathbf{I}_{12}$ are constant, r and $\mathbf{E}_{12}$ are variable.

FIG. 5b. Locus of the terminus of the variable voltage vector $\mathbf{E}_{12}$.

The result of expanding Eq. (23) and equating reals and imaginaries is:

$$\left. \begin{aligned} rI_1 - X_L I_2 &= e_1 \quad (a) \\ rI_2 + X_L I_1 &= e_2 \quad (b) \end{aligned} \right\} \quad (24)$$

Multiplying Eq. (24a) by I_2 and (24b) by I_1 and subtracting we have:

$$e_2 I_1 - e_1 I_2 - X_L I^2 = 0 \quad (25)$$

This is the equation of a straight line (Fig. 5b) intersecting the R - and j -axes respectively at the points:

$$\left. \begin{aligned} e'_1 &= -X_L I^2 / I_2 \quad (a) \\ e'_2 &= X_L I^2 / I_1 \quad (b) \end{aligned} \right\} \quad (26)$$

Dividing Eq. (26a) by (26b) we have:

$$(e'_1 / e'_2) = I_1 / I_2 \quad (27)$$

Eq. (27) shows that the intercepts of the voltage vector on the R - and j -axes are respectively proportional to the real- and j -components of $\mathbf{I}_{12}$. The locus of the terminus of $\mathbf{E}_{12}$ is therefore parallel to the current vector $\mathbf{I}_{12}$ (see Fig. 5b).

Physically the voltage cannot lead the current by more than 90° . Therefore the locus starts at the terminus of $\mathbf{E}_0$ and extends to infinity as shown in Fig. 5b.

This locus shows that the voltage required to send a constant current $\mathbf{I}_{12}$ through the circuit Fig. 5a is a minimum when the voltage vector $\mathbf{E}_0$ is perpendicular to $\mathbf{I}_{12}$. This is true because the impedance is a minimum when $r = 0$. Furthermore when $r = 0$ the circuit consists only of X_L and therefore the phase angle between the two vectors is 90° . Now as the resistance of the circuit increases the voltage has to increase in order to maintain the vector current constant. Also the phase angle ϕ becomes less than 90° . This is demonstrated by the vector $\mathbf{E}_{12}$, Fig. 5b. When r is infinite $\mathbf{E}_{12}$ also becomes infinite.

CASE b_3 : $\mathbf{I}_{12}$ CONSTANT; r , x_L AND $\mathbf{E}_{12}$ VARIABLE. — When r and x_L (Fig. 6a) vary the locus becomes indefinite. The locus, however, is definite if the mode of simultaneous variation of r and x_L is specified.

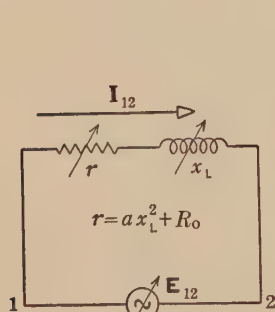


Fig. 6a

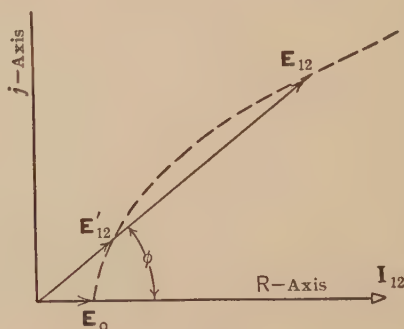


Fig. 6b

FIG. 6a. A series circuit in which $\mathbf{I}_{12}$ is constant, r , x_L , and $\mathbf{E}_{12}$ are variable.

FIG. 6b. Locus of the terminus of the variable voltage vector $\mathbf{E}_{12}$ when r and x_L so vary that $r = ax_L^2 + R_0$.

As an illustration consider such a variation of the circuit characteristics that:

$$r = ax_L^2 + R_0 \quad (28)$$

For simplicity let the current be taken along the R -axis. Then the current-voltage relation is expressed by the equation:

$$(r + jx_L)(I + j0) = e_1 + je_2 \quad (29)$$

Expanding Eq. (29) and equating reals and imaginaries we have:

$$\left. \begin{aligned} r &= e_1/I & (a) \\ x_L &= e_2/I & (b) \end{aligned} \right\} \quad (30)$$

Substituting Eq. (28) in (30a) and solving for x_L we have:

$$x_L^2 = (e_1 - R_0 I)/(aI) \quad (31)$$

Squaring Eq. (30b) and equating it to (31) we have:

$$(e_2^2/I^2) = (e_1 - R_0 I)/(aI) \quad (32)$$

whence

$$Ie_1 - ae_2^2 - R_0 I^2 = 0 \quad (33)$$

This is the equation of a parabola (Fig. 6b) with vertex at $\mathbf{E}_0 = R_0 I_{L2}$.

If, in Eq. (28), $a = 0$ then $r = R_0 = \text{constant}$, which is case b_1 . Substituting $a = 0$ in Eq. (33) we find that the locus becomes a straight line parallel to the j -axis and cutting the R -axis at the point $e'_1 = R_0 I$. This is case b_1 discussed above.

3. Given the Locus Required the Mode of Variation of the Circuit Characteristics. — In practice the locus is sometimes determined by test and it is required to find the mode of simultaneous variation of the circuit characteristics which produces such a locus. In performing such tests either the voltage is kept constant and the current allowed to vary, or the current is maintained constant and the voltage varied. Hence this topic divides itself into the following two cases:

(A) voltage constant, current variable.

(B) voltage variable, current constant.

On account of the similarity in the treatment of these two cases only the first will be dealt with in the text. The second will be reserved as an exercise to the student.

CASE (A): VOLTAGE CONSTANT, CURRENT VARIABLE. — The operation of a synchronous motor during its starting period furnishes a practical example of this case. Here a constant voltage is applied to the machine and simultaneous readings of voltage current and power input are taken at very short intervals of time. When the test is completed a vector diagram is drawn showing the constant voltage vector and the supposedly steady* and variable current vectors. Let a smooth curve be drawn joining the termini of the current vectors and let the equation of this curve be expressed either in rectangular or in polar coördinates. Thus:

$$\left. \begin{aligned} i_1 &= f(i_2) & (a) \\ I &= F(\beta) & (b) \end{aligned} \right\} \quad (34)$$

* "Steady" is here used as an antonym to "transient."

I and β are the variable modulus and argument of the current vector, while i_1 and i_2 are its components along the R - and j -axes respectively.

The problem is to determine the mode of variation, of the characteristics of the machine, which gives rise to this current locus.

Solution (a): Using Rectangular Coördinates. — In so far as the current-voltage relation is concerned any electrical machine may be represented by an equivalent series or parallel circuit consisting of r , $\mathfrak{L}$, and C . Hence either of the following equations holds true:

$$\left. \begin{aligned} (i_1 \pm ji_2)(r \pm jx) &= E_1 + jE_2 & (a) \\ (i_1 \pm ji_2) &= (g \mp jb)(E_1 + jE_2) & (b) \end{aligned} \right\} \quad (35)$$

These expressions could be solved for i_1 and i_2 or for I and β in terms of the circuit characteristics and the constant impressed voltage. Thus by expanding Eq. (35a), and equating reals and imaginaries, we obtain:

$$\left. \begin{aligned} i_1 &= (E_1r + E_2x)/(r^2 + x^2) & (a) \\ i_2 &= (E_2r - E_1x)/(r^2 + x^2) & (b) \end{aligned} \right\} \quad (36)$$

On the other hand, Eq. (35b) gives (by expanding and equating reals and imaginaries) the following values:

$$\left. \begin{aligned} i_1 &= gE_1 + bE_2 & (a) \\ i_2 &= gE_2 - bE_1 & (b) \end{aligned} \right\} \quad (37)$$

Substituting the values of i_1 and i_2 as expressed by Eqs. (36) or (37) in Eqs. (35) we obtain equations in r and x or in g and b respectively. From these equations, r can be determined as a function of x , or g as a function of b .

As a simple illustration let the locus of the variable current, as determined by test, be the ellipse shown in Fig. 7b, and let the equation of this locus be:

$$i_1^2 + 4i_2^2 = I_0^2 = G_0^2 E^2 \quad (38)$$

It is required to determine a parallel circuit (Fig. 7a) with such variable characteristics that the total current, flowing through it, shall have this locus.

Since the voltage vector is taken along the reference axis (see Fig. 7b), $E_2 = 0$. Therefore Eqs. (37) reduce to the form $i_1 = gE$ and $i_2 = -bE$. Substituting these values of i_1 and i_2 in Eq. (38) and solving for g we have:

$$g = \sqrt{G_0^2 - 4b^2} \quad (39)$$

Hence a parallel circuit with characteristics which vary according to Eq. (39) gives a variable current whose locus is the ellipse shown in Fig. 7b.

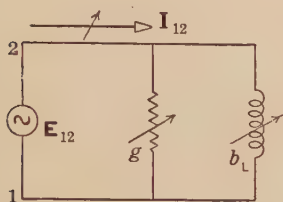


Fig. 7a

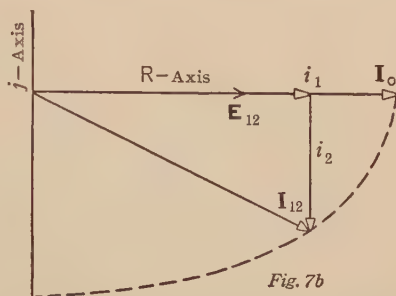


Fig. 7b

FIG. 7a. A parallel circuit in which E_{12} is constant, g , b_L , and I_{12} are variable.

FIG. 7b. Locus of the terminus of the variable current vector I_{12} as determined by test.

Solution (b): Using Polar Coördinates.—The equations, in polar form, expressing the current voltage relations in a circuit with R , $\mathcal{L}$, and C , are:

$$\left. \begin{aligned} I e^{j\beta} z e^{\pm j\phi} &= E e^{j\alpha} & \therefore I z e^{j(\beta \pm \phi)} &= E e^{j\alpha} & (a) \\ I e^{j\beta} &= y e^{\pm j\phi} E e^{j\alpha} & \therefore I e^{j\beta} &= y E e^{j(\alpha \pm \phi)} & (b) \end{aligned} \right\} \quad (40)$$

From Eq. (40b) we have

$$\left. \begin{aligned} \beta &= (\alpha \pm \phi) & (a) \\ I &= y E & (b) \end{aligned} \right\} \quad (41)$$

If the functional relation between I and β (see Eq. 34) be known from test then the corresponding relation between y and ϕ can be determined by the use of Eqs. (41).

As an illustration consider again Fig. 7b. The equation of the ellipse in polar form is:

$$I^2 (\cos^2 \beta + 4 \sin^2 \beta) = G_0^2 E^2 \quad (42)$$

Remembering that E is taken along the reference axis (that is, $\alpha = 0$) and substituting Eqs. (41) in (42) we have:

$$y^2 (\cos^2 \phi + 4 \sin^2 \phi) = G_0^2 \quad (43)$$

But from the admittance triangle (Fig. 2, Chapter VIII):

$$\cos \phi = g/y \quad \sin \phi = b/y \quad (44)$$

* Eq. (42) is obtained by simply expressing i_1 and i_2 in terms of the modulus and argument of the vector I , namely, $i_1 = I \cos \beta$, and $i_2 = I \sin \beta$, and substituting these values in Eq. (38).

Substituting Eqs. (44) in (43) and solving for g we have:

$$g = \sqrt{G_0^2 - 4b^2} \quad (45)$$

This result is identical with that given in Eq. (39).

4. Conclusion. — Possibly one of the most interesting topics of the theory of electric circuits is the study of the current and voltage relations when the characteristics of the circuit are variable. An insight into the vastness of the subject may be had by reviewing cases a_3 and b_3 of Art. 2. Thus what would be the locus if $x_L = \epsilon'$, or if $r = ax_L^2 + bx_L + R_0$ or in general if $r = f(x_L)$ or $x_L = F(r)$?

Again, the behavior of electrical machinery, and electric circuits in general, may be such as to give current or voltage loci that are neither a circle nor a straight line. It is very convenient, for purposes of study or analysis, to represent such circuits or machines by a simple electric circuit which behaves identically as the machine itself.

It has been the author's intention to present a method of attack which possess both simplicity and generality. This is made feasible through the use of complex quantities.

PROBLEMS

1. Give some cases, in engineering practice, where the circuit characteristics are variable.

2a. Show that if a , b , and c are constant and w , x , y are variable then a circle can be represented by equations of the form:

$$(x + jy)(c + jw) = a + jb$$

$$(x + jy)(w + jc) = a + jb$$

Give, for both equations, the magnitude of the diameter and the point at which the center of the circle lies.

2b. Prove that a straight line may be represented by equations of the form:

$$(x + jy) = (a + jb)(c + jw)$$

$$(x + jy) = (a + jb)(w + jc)$$

Give the intercepts of the straight line on the x - and y -axes respectively.

2c. A series circuit consists of r and x_c and a constant impressed voltage E . What is the locus of the terminus of the variable current vector when:

(1) R is constant and x_c is variable.

(2) r is variable and X_c is constant.

(3) r and x_c so vary that $x = 2r^2 + 3$.

Hint. — In part (3) assume for simplicity that the voltage vector $= 0 + jE$ and if the equation of the locus is such that you can not tell what the curve is then plot the locus.

2d. A series circuit, consisting of r and x_c , has a constant vector current $\mathbf{I}$ flowing through it. Draw the locus of the terminus of the variable voltage vector when:

- (1) R is constant and x_c is variable.
- (2) r is variable and X_c is constant.
- (3) r and x_c so vary that $r = x_c^2 + 2x_c + 3$.

Hint. — In part (3) assume for simplicity that the current vector is $I + j0$, and plot the locus.

2e. A series circuit consists of r , x_L , and x_c and a constant impressed voltage $\mathbf{E}$. Give the locus of the variable vector current when:

- (1) R is constant and $(x_L - x_c)$ is variable.
- (2) r is variable and $(X_L - X_c)$ is constant such that (a) $X_L > X_c$ and (b) $X_L < X_c$.
- (3) Both r and $(x_L - x_c)$ are variable such that $r = 2(x_L - x_c)^2 + 5$.

Note. — See hint, problem 2c.

2f. A series circuit, consisting of r , x_L , and x_c , has a constant vector current $\mathbf{I}$ flowing through it. What would be the locus of the terminus of the variable voltage vector when:

- (1) R is constant and $(x_L - x_c)$ is variable.
- (2) r is variable and $(X_L - X_c)$ is constant such that (a) $X_L > X_c$ and (b) $X_L < X_c$.
- (3) Both r and $(x_L - x_c)$ are variable such that $(x_L - x_c) = (2r - 5)/(3r + 1)$.

Note. — See hint, problem 2d.

2g. A parallel circuit consists of a conductance g , an inductive susceptance b_L , and a constant impressed voltage $\mathbf{E}$. Determine the locus of the variable current vector when:

- (1) G is constant and b_L is variable.
- (2) g is variable and B_L is constant.
- (3) Both g and b_L are variable such that $g = 1/b_L$.

Note. — See hint, problem 2c.

2h. A parallel circuit comprising g and b_L has a constant current $\mathbf{I}$ flowing through it. Draw the locus of the terminus of the variable voltage vector when:

- (1) G is constant and b_L is variable.
- (2) g is variable and B_L is constant.
- (3) Both g and b_L are variable such that $b_L = 1/(g + 6)$.

Note. — See hint, problem 2d.

2i. A constant voltage $\mathbf{E}$ is impressed on a parallel circuit consisting of g and b_c . What would be the locus of the vector current when:

- (1) G is constant and b_c is variable.
- (2) g is variable and B_c is constant.
- (3) Both g and b_c are variable such that $g = b_c^2 + 2b_c + 1$.

Note. — See hint, problem 2c.

2j. The current vector, in a parallel circuit of g and b_c , is maintained constant. Determine the locus of the variable voltage vector when:

- (1) G is constant and b_c is variable.
- (2) g is variable and B_c is constant.
- (3) Both g and b_c are variable such that $b_c = g^2 + 3g + 5$.

Note. — See hint, problem 2d.

2k. A constant voltage is impressed on a parallel circuit consisting of g , b_L , and b_c . Determine the locus of the variable vector current when:

- (1) G is constant and $(b_L - b_c)$ is variable.
- (2) g is variable and $(B_L - B_c)$ is constant such that (a) $B_L > B_c$ and (b) $B_c > B_L$.
- (3) Both g and $(b_L - b_c)$ are variable such that $g = (b_L - b_c)^2 + 3$.

Note. — See hint, problem 2c.

2l. The current vector, in a parallel circuit of g , b_L , and b_c , is maintained constant. Draw the locus of the variable voltage vector when:

- (1) G is constant and $(b_L - b_c)$ is variable.
- (2) g is variable and $(B_L - B_c)$ is constant such that (a) $B_L > B_c$ and (b) $B_c > B_L$.
- (3) Both g and $(b_L - b_c)$ are variable such that:

$$g = [2(b_L - b_c) + 4]/[3(b_L - b_c) + 9]$$

Note. — See hint, problem 2d.

2m. Three impedance coils having resistances $r_1 = 2$, $r_2 = 3$, and $r_3 = 4$ ohms, and inductances $\mathcal{L}_1 = 13$, $\mathcal{L}_2 = 10$, and $\mathcal{L}_3 = ?$ mh are connected in parallel across a 220-volt, 60-cycle source. If the total current is 110 amperes, what are the inductance of the third coil and the phase angle of the total current?

2n. Referring to Fig. 6b it will be observed that there exist two values of the variable voltage vector $\mathbf{E}_{12}$ at which the power factor of the circuit is the same. These values are $\mathbf{E}_{12}$ and $\mathbf{E}'_{12}$. Explain!

2o. A resistance R and an inductance $\mathcal{L}$ are connected and placed inside a sealed box. Devise an experimental method of ascertaining whether the connection is series or parallel without opening the box. Give reasons for the validity of your method.

3a. The locus of the variable voltage, which maintains a constant current in a circuit, is of the form

$$4e^2_1 - 9e^2_2 = IR_0$$

Express the resistance of the circuit as a function of its reactance assuming the constant current $\mathbf{I}$ to be $\mathbf{I} + j0$.

3b. The current locus in a circuit (parallel or series) is of the form:

$$i^2_1 - i^2_2 = E^2g^2_0$$

Assuming the constant voltage to be $\mathbf{E}_{12} = 0 + jE$, what is the functional relation between the variable circuit characteristics which gives rise to such a current locus?

3c. The voltage locus in a circuit is of the form:

$$e^2_2 = 4e_1 + 8$$

Assuming the constant current to be $\mathbf{I}_{12} = 5 + j0$ what is the mode of simultaneous variation of the circuit characteristics?

3d. Check the answers to problems 3a, 3b, and 3c by solving these problems in the polar form.

CHAPTER X

SERIES-PARALLEL COMBINATIONS OF CIRCUIT CHARACTERISTICS AND OF ALTERNATING EMF. SOURCES

1. The Concept of Equivalence. — If the characteristics of a series and a parallel circuit are so selected that oscillographic records of the impressed voltage and total current for the series circuit are identical with those for the parallel circuit, then the two circuits are said to be equivalent. Thus let a resistance R and an inductance X_L (Fig. 1a) be connected in series across a source of sine voltage $\mathbf{E}_{12}$ and let a two-element oscillograph be so connected as to register simultaneously the sine waves e and i (Fig. 1c). Next, let a conductance G and a susceptance B_L (Fig. 1b) be connected in parallel and subjected to the same

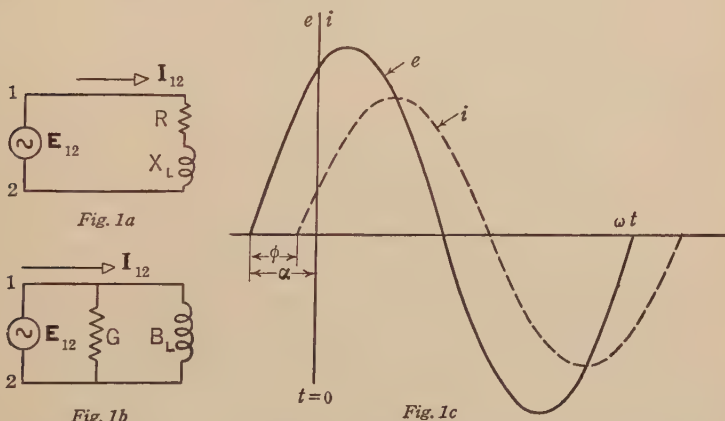


FIG. 1a. A series circuit consisting of R , X_L , and a sine Emf. $\mathbf{E}_{12}$.

FIG. 1b. A parallel circuit which is equivalent to the series circuit Fig. 1a.

FIG. 1c. Sine waves in the circuits Figs. 1a and 1b.

sine Emf. $\mathbf{E}_{12}$, and let a two-element oscillograph be so connected as to register e and i . If the two oscillographic records are identical in every respect (see Fig. 1c) then the two circuits are said to be equivalent. The identity of the two records implies the following conditions:

(a) That the frequency and amplitude of the sine voltage impressed on the two circuits shall be the same.

(b) That the phase angle ϕ between the voltage and current waves shall be the same for the parallel as well as for the series circuit.

(c) That the amplitude of the total sine current flowing through the two circuits shall be the same

These three physical conditions may be reduced to two if they are stated in vectorial language. Thus let the subscript s stand for series and the subscript p stand for parallel. Then the foregoing conditions become:

(a) The impressed voltage vectors $\mathbf{E}_s$ and $\mathbf{E}_p$ are identical.

(b) The total-current vectors $\mathbf{I}_s$ and $\mathbf{I}_p$ are equal.

2. Mathematical Conditions for Equivalence. — The two vector conditions for equivalence given above may be stated mathematically thus:

$$\left. \begin{aligned} \mathbf{E}_s &= \mathbf{E}_p & (a) \\ \mathbf{I}_s &= \mathbf{I}_p & (b) \end{aligned} \right\} \quad (1)$$

But:

$$\left. \begin{aligned} \mathbf{E}_s &= Z\mathbf{I}_s & (a) \\ \mathbf{E}_p &= \mathbf{I}_p/Y & (b) \end{aligned} \right\} \quad (2)$$

Dividing Eq. (2a) by Eq. (2b) and substituting $\mathbf{E}_s$ and $\mathbf{I}_s$ for $\mathbf{E}_p$ and $\mathbf{I}_p$, respectively, we have:

$$ZY = 1 \quad (3)$$

whence

$$\left. \begin{aligned} Y &= 1/Z & (a) \\ Z &= 1/Y & (b) \end{aligned} \right\} \quad (4)$$

where

$$\left. \begin{aligned} Y &= G \mp jB & (a) \\ Z &= R \pm jX & (b) \end{aligned} \right\} \quad (5)$$

It will be noted that no limitations have been placed on the physical properties of the characteristics dealt with. In fact B and X in Eqs. (5) may be inductive or capacitive or both. Since X and B are functions of the frequency of the voltage source, two circuits that are equivalent for a certain frequency will not be equivalent for any other frequency. This is an important fact which we shall use to show that the concept of equivalence holds true only for circuits subjected to sine voltages but becomes meaningless when non-sine Emfs. are used.

3. Series-Parallel Combinations of Circuit Characteristics and Their Equivalent Series or Parallel Circuits. — We have heretofore discussed two types of circuits, namely: (a) the series circuit in which the characteristics are so connected that the same current flows through them and (b) the parallel circuit in which the characteristics are so arranged that they are across the same impressed Emf. Now it is

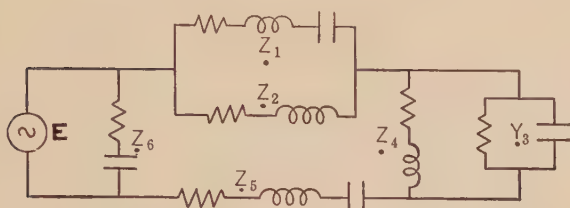


FIG. 2. A series parallel circuit.

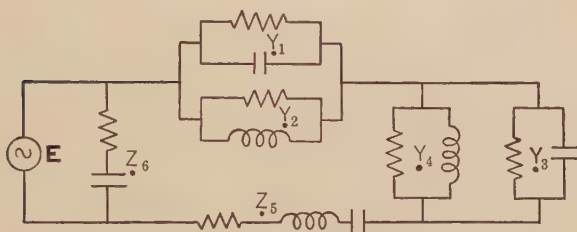


Fig. 2a

FIG. 2a. First step in converting the circuit, Fig. 2, into an equivalent series circuit.

$$Y_1 = (1/Z_1); \quad Y_2 = (1/Z_2); \quad Y_4 = (1/Z_4)$$

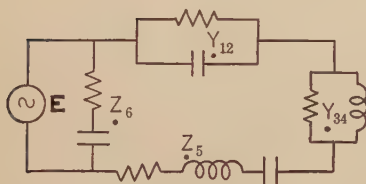


Fig. 2b

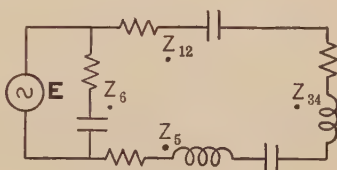


Fig. 2c

FIGS. 2b and 2c. Second and third steps in converting the circuit, Fig. 2, into an equivalent series circuit.

$$2nd \text{ step (Fig. 2b)} \quad Y_{12} = Y_1 + Y_2; \quad Y_{34} = Y_3 + Y_4$$

$$3rd \text{ step (Fig. 2c)} \quad Z_{12} = (1/Y_{12}); \quad Z_{34} = (1/Y_{34})$$

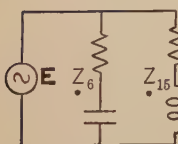


Fig. 2d

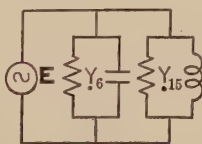


Fig. 2e

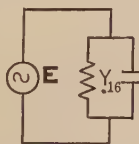


Fig. 2f



Fig. 2g

FIGS. 2d to 2g. Fourth to seventh steps in converting the circuit, Fig. 2, into an equivalent series circuit.

$$4th \text{ step (Fig. 2d)} \quad Z_{15} = (Z_{12} + Z_{34} + Z_5)$$

$$5th \text{ step (Fig. 2e)} \quad Y_{15} = (1/Z_{15}); \quad Y_6 = (1/Z_6)$$

$$6th \text{ step (Fig. 2f)} \quad Y_{16} = (Y_{15} + Y_6)$$

$$7th \text{ step (Fig. 2g)} \quad Z_{16} = (1/Y_{16})$$

conceivable that a combination of these two types may exist in the same circuit. Fig. 2 illustrates such a combination, which is known as a series-parallel circuit. These circuits are commonly met with in electrical engineering practice. A network of power lines, or even an electrical machine (for example, the transformer) may be represented by a series-parallel circuit.

Now, for some purposes (as computing the total current supplied by the source) it is convenient to convert a circuit, like that shown in Fig. 2, into an equivalent series or parallel circuit. This is accomplished by converting successively the individual series-parallel branches into a pure series or a pure parallel branch by the use of Eqs. (4). The operation is greatly simplified and errors are avoided if the various steps in the conversion are outlined by connection diagrams as is shown in Figs. 2a to 2g.

The following rules must be observed in converting a series-parallel circuit to an equivalent circuit:

- (a) Admittances in parallel can be added only if the potential drop across them is the same.
- (b) Impedances in series can be added only if the same current flows through them.
- (c) Never add Y and Z . They have different physical dimensions. Adding them is like adding houses and trees.

The first two of these rules follow directly from the definitions of a series and a parallel circuit. The last is just common sense.

The student is strongly urged to study the successive steps of conversion, outlined in Figs. 2a to 2g, in the light of these rules and to make clear to himself how each step is attained. Thus Fig. 2a is obtained by converting Z_1 , Z_2 , and Z_4 into $Y_1 = 1/Z_1$, $Y_2 = 1/Z_2$, and $Y_4 = 1/Z_4$. Referring next to Fig. 2b, we note that Y_1 and Y_2 have been added giving Y_{12} . Also Y_3 and Y_4 have been combined into a single admittance, Y_{34} . It is needless to explain each step. The student can now follow the process by studying the successive figures. Indeed a thorough study of these figures is of more value to the reader than any explanation that the author may give in an effort to elucidate the subject.

4. Non-Convertible Combinations of Circuit Characteristics. — It must not be inferred from the foregoing treatment that all impedance combinations are reducible to an equivalent series or an equivalent parallel circuit. Indeed there exist certain combinations which cannot be converted to a simple series or parallel circuit because certain impedances are so connected that none of the branches is a pure series or a pure parallel branch. The circuit shown in Fig. 3 is an example of this type.

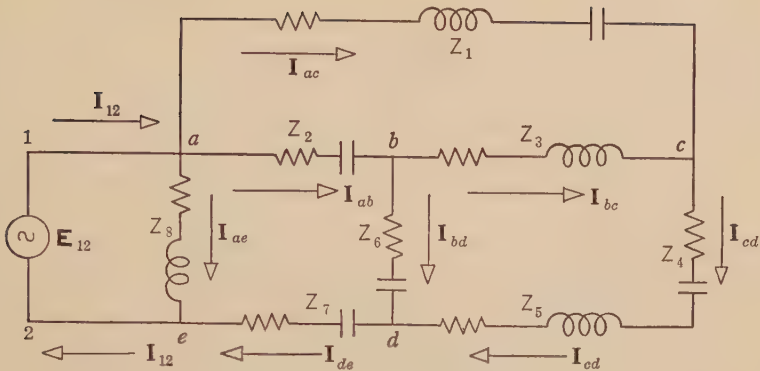


FIG. 3. A non-convertible combination of circuit characteristics.

5. Resonance in Series-Parallel Circuits.—The conditions for resonance in a series or a parallel circuit are given in Chapters VII and VIII. These are respectively:

$$\left. \begin{aligned} X_L &= X_c & (a) \\ B_L &= B_c & (b) \end{aligned} \right\} \quad (6)$$

From these conditions the resonant frequency is determined and defined as:

$$f = 1/(2\pi\sqrt{LC}) \quad (7)$$

Since Eqs. (6) define the condition for resonance in a pure series or a pure parallel circuit, they are general and apply whenever a pure circuit is considered. The question which arises is whether Eq. (7) is also general, that is, whether it applies to an equivalent circuit that has been obtained through the conversion of another circuit.

In order to answer this question it will be necessary first to compute the values of X and B in the equivalent series or equivalent parallel circuit. This can be accomplished by expanding Eqs. (4) and equating reals and imaginaries. We thus have:

$$\left. \begin{aligned} G \mp jB &= 1/(R \pm jX) = (R \mp jX)/(R^2 + X^2) & (a) \\ R \pm jX &= 1/(G \mp jB) = (G \pm jB)/(G^2 + B^2) & (b) \end{aligned} \right\} \quad (8)$$

whence

$$\left. \begin{aligned} G &= R/(R^2 + X^2); & B &= X/(R^2 + X^2) & (a) \\ R &= G/(G^2 + B^2); & X &= B/(G^2 + B^2) & (b) \end{aligned} \right\} \quad (9)$$

Now Eqs. (9) show that the equivalent reactance and susceptance have values which are functions of the conductance and resistance respectively. But Eq. (7) is independent of R and G . Therefore Eq. (7)

does not generally hold true for a circuit that has been obtained through conversion.

In order to determine the resonant frequency of a series parallel circuit, first convert it to an equivalent series or parallel circuit; then apply

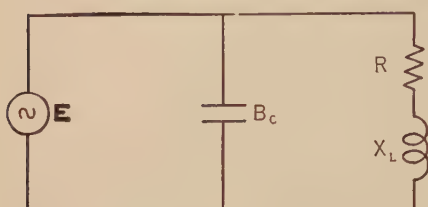


FIG. 4. Resonance in a series-parallel circuit.

Eqs. (6) to the equivalent circuit and solve for f . This will give the frequency for voltage or current resonance according as to whether the circuit has been converted to an equivalent series or an equivalent parallel circuit respectively.

Consider as an illustration the circuit shown in Fig. 4.

Let it be required to determine the frequency at which current resonance exists.

The equivalent admittance of the series branch is:

$$Y_L = G - jB_L = (R/Z_L^2) - \{jX_L/[R^2 + (2\pi f\mathcal{L})^2]\} \quad (10)$$

The condition for current resonance is, according to Eq. (6b), $B_c = B_L$. Equating B_c to B_L (as determined from Eq. 10) we have:

$$2\pi fC = 2\pi f\mathcal{L}/[R^2 + (2\pi f\mathcal{L})^2] \quad (11)$$

whence
$$f = 1/[2\pi \sqrt{(1/\mathcal{L}C) - (R^2/\mathcal{L}^2)}] \quad (12)$$

which is the frequency at which current resonance occurs.

In a complicated circuit, such as that shown in Fig. 2, there exists more than one resonant frequency. In fact any branch, comprising $\mathcal{L}$ and C , exhibits either voltage or current resonance at a certain frequency. Thus the impedance Z_1 (Fig. 2) shows voltage resonance at a frequency determined by the condition $X_{L1} = X_{C1}$. Moreover the combined admittances Y_1 and Y_2 (Fig. 2a) exhibit current resonance at such a frequency that $B_{C1} = B_{L1} + B_{L2}$. Here is a case where two distinct frequencies exist, the one resulting in voltage and the other in current resonance.

6. Series Emf. Combinations (Kirchhoff's Emf. Law). — The circuit, Fig. 5a, consists of two transformers with terminal voltages E_{12} and E_{23} . These transformers are connected in series to obtain a high voltage for dielectric testing. It is required to determine the voltage E_{13} which is impressed on the test impedance Z .

The Emf. E_{12} (see Art. 4, Chapter VI) sends a current through the

external circuit from point 1 toward point 2 (in the direction of I_{13}). Similarly E_{23} sends a current in the same direction. Hence the two Emfs. are additive and we have (by Kirchhoff's Emf. law):

$$E_{13} = E_{12} + E_{23} = ZI_{13} \quad (13)$$

The vector sum of E_{12} and E_{23} is represented graphically by Fig. 5b.

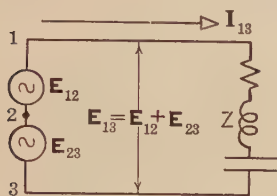


Fig. 5a

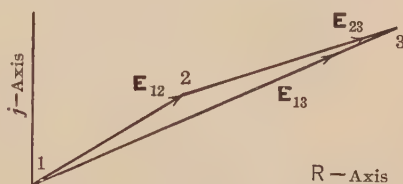


Fig. 5b

FIG. 5a. A series circuit consisting of an impedance Z and two sine Emfs. E_{12} and E_{23} .

FIG. 5b. Vector diagram of Emfs. in the circuit Fig. 5a.

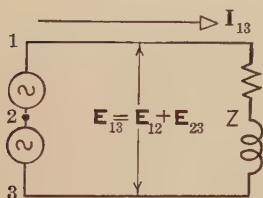


Fig. 6a

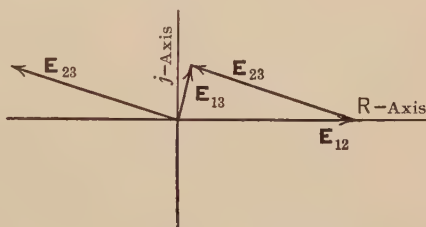


Fig. 6b

FIG. 6a. A series circuit consisting of an impedance Z and two sine Emfs. E_{12} and E_{23} .

FIG. 6b. Vector diagram of the Emfs. in the circuit Fig. 6a.

Next consider the circuit Fig. 6a. Let this circuit represent one phase of an alternator supplying power, to one phase of a synchronous motor, through the impedance Z . As explained above, both Emfs. tend to send a current through the impedance Z in the direction of I_{13} . Hence their sum is expressed by Eq. (13).

A comparison of the vector diagrams, Figs. 5b and 6b, shows that although the net Emf. in Fig. 5b is greater than the Emf. of either source, such is not the case in Fig. 6b. This is due to the fact that E_{23} in this latter case is almost opposite to E_{12} . Indeed a synchronous motor has an induced counter Emf. which is almost equal and opposite to the applied Emf.

In general if, in going around a closed circuit, the terminals of the

Emf. sources are numbered consecutively 1, 2, 3 . . . n , then the resultant Emf. is:

$$E_{1n} = E_{12} + E_{23} + \cdots + E_{(n-1)n}^* \quad (14)$$

7. Emf.-Impedance Combinations. — It is shown in Chapters VII and VIII that Kirchhoff's Laws (for circuits containing sine Emfs. and currents) may be stated as follows:

(1) The vector sum of the Emfs. around any closed circuit is equal to the vector sum of the IZ drops.

(2) The vector sum of the currents flowing away from a point is equal to the vector sum of the currents flowing toward that point.

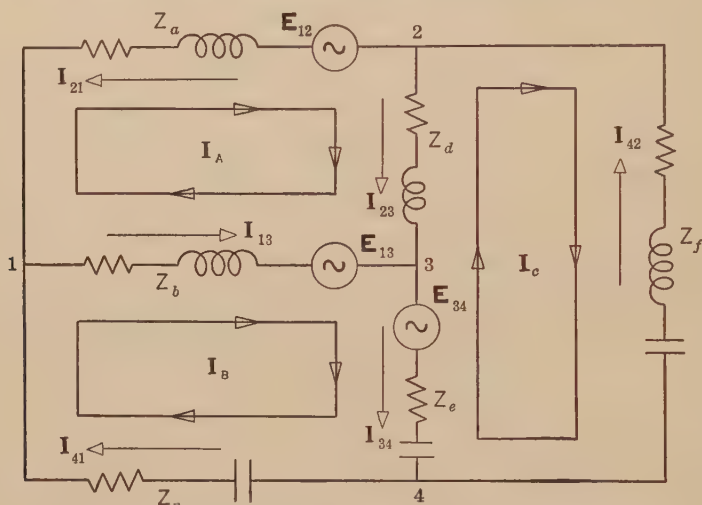


Fig. 7. A circuit consisting of several Emfs. and impedances.

Before applying these two laws to the solution of a circuit (Fig. 7) containing several Emfs. and impedances it will be necessary to introduce the following definitions:

(a) A *Loop* is any closed path within a circuit. Thus in Fig. 7 the closed path 1 2 3 1 is a loop.

(b) A *Junction* is any point in a circuit which is common to two or more loops. In Fig. 7 the points 1, 2, 3, and 4 are junctions.

(c) A *Branch* is any connection between two junctions. This connection may consist of impedances, of Emfs. or of both. In Fig. 7 the impedance Z_a and the Emf. E_{12} form a branch between the junctions 1 and 2.

* It is to be understood that the numbering and the sequence of subscripts in Eq. (14) conforms with the established conventions explained in Art. 4c, Chapter VI, relative to oscillograph connections.

Three problems may arise in dealing with circuits similar to that shown in Fig. 7:

- (A) The Emfs. and the impedances are given and the currents in the various branches are required.
- (B) The Emfs. and currents are known and the impedances sought.
- (C) The currents and impedances are given and the Emfs. are demanded.

We shall discuss each of these problems:

(A) *Given Emfs. and Impedances; Required the Branch Currents.* — This problem can be solved by writing Kirchhoff's current and Emf. laws which yield enough independent equations to determine the unknown quantities.

Assume the currents (Fig. 7) to be directed as shown.* Applying Kirchhoff's current law to junctions 1, 2, 3, and 4 respectively, we have:

$$\left. \begin{aligned} \mathbf{I}_{21} + \mathbf{I}_{41} &= \mathbf{I}_{13} & (a) \\ \mathbf{I}_{42} &= \mathbf{I}_{21} + \mathbf{I}_{23} & (b) \\ \mathbf{I}_{23} + \mathbf{I}_{13} &= \mathbf{I}_{34} & (c) \\ \mathbf{I}_{34} &= \mathbf{I}_{41} + \mathbf{I}_{42} & (d) \end{aligned} \right\} \quad (15)$$

Applying Kirchhoff's Emf. law to the loops 1 2 3 1, 1 3 4 1, 2 3 4 2, and 1 2 3 4 1 we obtain respectively:

$$\left. \begin{aligned} \mathbf{E}_{12} - \mathbf{E}_{13} &= \mathbf{Z}_a \mathbf{I}_{21} + \mathbf{Z}_b \mathbf{I}_{13} - \mathbf{Z}_d \mathbf{I}_{23} & (a) \\ \mathbf{E}_{13} + \mathbf{E}_{34} &= -\mathbf{Z}_b \mathbf{I}_{13} - \mathbf{Z}_c \mathbf{I}_{41} - \mathbf{Z}_e \mathbf{I}_{34} & (b) \\ \mathbf{E}_{34} &= -\mathbf{Z}_c \mathbf{I}_{34} - \mathbf{Z}_d \mathbf{I}_{23} - \mathbf{Z}_f \mathbf{I}_{42} & (c) \\ \mathbf{E}_{12} + \mathbf{E}_{34} &= \mathbf{Z}_a \mathbf{I}_{21} - \mathbf{Z}_c \mathbf{I}_{41} - \mathbf{Z}_e \mathbf{I}_{34} - \mathbf{Z}_d \mathbf{I}_{23} & (d) \end{aligned} \right\} \quad (16)^\dagger$$

Now let the Emfs. and impedances be given and the currents required. Only six independent equations will be needed because there are six unknown currents. Eqs. (15) and (16) give eight equations. We, therefore, conclude that either there exists no solution or two of the equations are dependent, that is, they can be derived from the other equations. Fortunately the latter is the case because any of Eqs. (15) can be derived by adding the remaining three equations. Therefore there really exist three independent expressions in Eq. (15). Similarly Eq. (16d) may be derived by adding Eqs. (16a) and (16b) and is dependent. We thus have six independent equations with six unknown currents. The simultaneous solution of these six independent equations (see Art. 12, Chapter I) gives the desired currents.

* The assumed direction of currents is immaterial.

† The student is apt to wonder why some of the ZI drops in Eqs. (16) bear negative signs. The reason is that the assumed direction of these currents is contrary to that dictated by the Emfs. Thus in loop 1 3 4 1 the Emf. sources tend to send currents in a direction opposite to that shown in Fig. 7. Hence all the currents in Eq. (16b) are negative.

The student is strongly urged to examine the equations for dependence before proceeding with the solution. This advice is given for two reasons:

(a) The inexperienced person is very apt to set up as many equations as there are unknown quantities but, owing to the fact that some of these equations are dependent, he would find himself unable to solve the problem.

(b) Much time is lost in duplicate algebraic processes if the student has more than the desired number of equations.

Generally speaking, independent Emf. equations will be had if these equations are written for all the single loops* occurring in the network.

The number of independent Emf. equations is equal to the number of single loops in the circuit. The number of independent current equations is equal to the number of junctions less one.

(B) and (C) *Given the Emfs. and Currents, Required the Impedances; or Given the Currents and Impedances Required the Emfs.* — The problem assumes a different aspect when the Emfs. and currents are given and it is required to find the impedances or when the currents and impedances are known and the applied Emfs. are sought. Since the current equations (see Eqs. 15) do not involve Z and E we are left with only the Emf. equations (Eqs. 16) from which to determine all the impedances or Emfs. Generally speaking there do not exist enough Emf. equations to determine the unknown Emfs. or impedances. The problem therefore becomes indeterminate. Physically this means that with a given set of vector currents and Emfs. it is possible to select an infinite number of impedances which would satisfy Kirchhoff's Emf. relations. Conversely with a given set of impedances and branch currents it is possible to select an infinite number of impressed Emfs. which would produce such branch currents in the given circuit.

In order to render the problem determinate it is necessary to specify the potentials of the junctions measured from some common reference point (say the earth) and to write Kirchhoff's Emf. equations in terms of those known potentials. Thus referring to Fig. (7) let the potentials of the junctions 1, 2, 3, and 4 measured from a common reference point 0 be E'_{01} , E'_{02} , E'_{03} , and E'_{04} respectively. Then:

$$\left. \begin{aligned} E'_{12} &= E'_{02} - E'_{01} = E_{12} - Z_a I_{21} & (a) \\ E'_{32} &= E'_{02} - E'_{03} = Z_d I_{23} & (b) \\ E'_{13} &= E'_{03} - E'_{01} = E_{13} + Z_b I_{13} & (c) \\ E'_{14} &= E'_{04} - E'_{01} = Z_c I_{41} & (d) \\ E'_{34} &= E'_{04} - E'_{03} = E_{34} + Z_c I_{34} & (e) \\ E'_{24} &= -Z_j I_{42} & (f) \end{aligned} \right\} \quad (17)$$

* A single loop is one which does not enclose any branches. Thus the loop (12341) in Fig. 7 is not a single loop because it encloses the branch 13.

We thus have six independent equations with six unknowns, either the unknown impedances or the unknown applied Emfs.*

8. Application of the Principle of Current Superposition to the Solution of a Circuit.—The solution of six simultaneous equations is tedious enough when such equations involve real numbers. With six simultaneous equations involving complex numbers the work becomes very irksome. In this article a method is outlined which cuts the number of simultaneous equations in half thereby simplifying the solution.

Since electricity behaves like an incompressible fluid electric currents are superposable, that is, a current flowing in one branch may be considered as the sum of two currents flowing in the same direction or the difference of two currents flowing in opposite directions. Thus, referring to Fig. 7, imagine three currents I_A , I_B , and I_C to circulate in the loops as shown. Then the currents in the various branches may be considered as made up of these three currents as follows:

$$\left. \begin{aligned} I_{21} &= -I_A & I_{23} &= I_A - I_C \\ I_{13} &= I_B - I_A & I_{34} &= I_B - I_C \\ I_{41} &= I_B & I_{42} &= -I_C \end{aligned} \right\} \quad (18)$$

Now let it be required to find the unknown six currents in the branches of Fig. 7. Only three independent equations are needed. These are the following Emf. equations for loops A , B , and C .

$$\left. \begin{aligned} E_{12} - E_{31} &= -Z_a I_A - Z_b(I_A - I_B) - Z_d(I_A - I_C) & (a) \\ E_{13} + E_{34} &= -Z_b(I_B - I_A) - Z_c I_B - Z_e(I_B - I_C) & (b) \\ E_{34} &= Z_d(I_C - I_A) + Z_e(I_C - I_B) + Z_f I_C & (c) \end{aligned} \right\} \quad (19)$$

Having computed I_A , I_B , and I_C one can evaluate the branch currents by using Eqs. 18.

The reader can appreciate the advantage of this method over the direct application of Kirchhoff's laws. The number of independent simultaneous equations here needed is equal to the number of single loops which the circuit comprises. In writing these equations one should be particularly careful in selecting the proper signs for the circulating currents. A study of Eqs. (19) in connection with Fig. 7 is here recommended.

PROBLEMS

1. The A.C. voltmeter and ammeter readings of two circuits are identical. Are these circuits necessarily equivalent? Why?

* In Fig. 7 there are only three applied Emfs. but the reader can easily imagine the same circuit as comprising one Emf. in each of the six branches.

2a. Determine the value of the conductance G and susceptance B_L which, when connected in parallel, will give a circuit equivalent to a series circuit having a resistance $R = 10$ ohms and a reactance $X_L = 5$ ohms.

2b. Give the numerical value of the resistance R and inductive reactance X_L which, when connected in parallel, will give a circuit equivalent to the series circuit specified in problem 2a.

2c. A series circuit consists of a resistance $R = 15$ ohms in series with a condenser of 1μ f. capacity and an inductance of 0.2×10^{-3} henry. Assuming that the frequency of the impressed voltage is 5000 cycles per second, determine a parallel combination that will be equivalent to the given series circuit.

2d. Let the resistance, condenser, and inductance specified in problem 2c be connected in parallel across a voltage source having a frequency of 5000 cycles per second. Find a series circuit that will be equivalent to this parallel combination.

2e. Prove, by numerical computations, that the characteristics for the equivalent parallel and series circuits of problems 2c and 2d will not be identical with the corresponding characteristics determined for a frequency of 10,000 cycles per second.

3. Find a series circuit which is equivalent to the series-parallel combination given in Fig. 8.

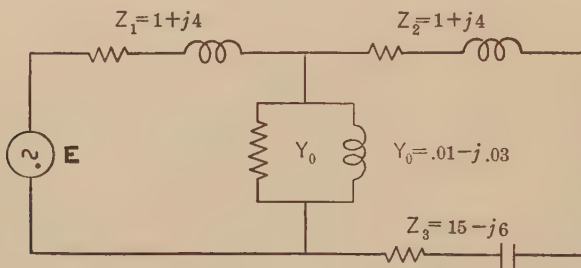


FIG. 8. A series-parallel circuit representing a transformer.

4a. Explain why the circuit shown in Fig. 3 is non-convertible.

4b. What change can be made in one of the connections of the impedance Z_1 (Fig. 4) in order to render the circuit convertible.

4c. Draw a non-convertible combination of impedances.

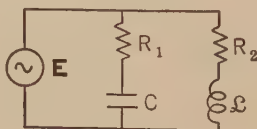


FIG. 9. A series-parallel circuit.

5a. Convert the series-parallel combination shown in Fig. 9 into a parallel circuit and determine its resonant frequency.

5b. Make a table comprising the following column headings: (a) branch, (b) characteristic, (c) type of resonance, (d) condition for resonance.

Now turn to Fig. 2 and fill out this table in the following manner:

(a) Branch	(b) Characteristic	(c) Type of resonance	(d) Condition for resonance
1	Z_1	voltage	$X_{L1} = X_{C1}$
1 and 2	Z_1 and Z_2	current	$B_{C1} = B_{L1} + B_{L2}$

Finally compute the resonant frequency for the first two rows.

6a. The secondary windings of three testing transformers are connected in series. If the secondary voltages of these transformers are respectively

$$\mathbf{E}_{12} = 220,000 \angle 35^\circ; \quad \mathbf{E}_{23} = 440,000 \angle 25^\circ \quad \text{and} \quad \mathbf{E}_{34} = 660,000 \angle 15^\circ;$$

what is the effective value of the terminal voltage $\mathbf{E}_{14}$?

6b. An 11,000-volt, 60-cycle alternator supplies power to a synchronous motor through an impedance $25 \angle 30^\circ$. If the voltage developed by the motor is 11,000 and the phase angle between the motor and alternator voltages is 170° , determine the effective value of the current which flows through the impedance.

7a. Write independent vector equation from which the unknown currents shown in Fig. 10 can be determined.

7b. Referring to Fig. 10 and problem 7a, let $\mathbf{E}_{01} = 110 \angle 0$, $\mathbf{E}_{02} = 110 \angle 120^\circ$, and $\mathbf{E}_{03} = 110 \angle 240^\circ$; and let $Z_{12} = 3 + j7$, $Z_{23} = 4 - j5$, and $Z_{31} = 5 + j3$. Determine the values of the unknown currents.

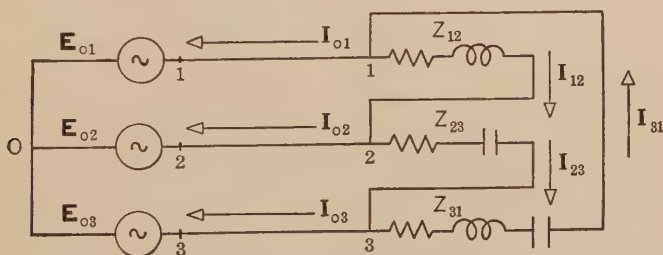


FIG. 10. A three-phase series parallel circuit.

7c. An A.C. bridge is a device used to measure R , $\mathcal{L}$, or C . The apparatus may assume various forms, among which is the one shown in Fig. 11. Let it be required to measure the impedance Z_1 . The characteristics Z_3 and Z_4 are varied until the galvanometer deflection is zero. Then knowing Z_2 , Z_3 and Z_4 , Z_1 may be computed. Develop a relation expressing the value of Z_1 in terms of the known characteristics.

Hint. — Make use of the facts (a) that the junctions 1 and 2 (Fig. 11) are at the same potential, (b) that the same current flows through Z_1 and Z_2 , and (c) that the same current flows through Z_3 and Z_4 .

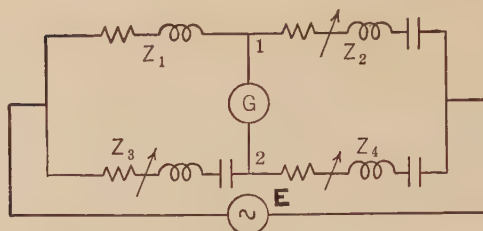


FIG. 11. A bridge circuit.

7d. Write independent equations from which the unknown currents (Fig. 3) can be determined. Do not solve these equations.

8a. Using the principle of superposition write four Emf. equations from which the circulating currents (Fig. 3) can be determined.

8b. Write equations expressing the branch currents (Fig. 3) in terms of the circulating currents.

8c. Let the impedances (Fig. 3) have the following values: $Z_1 = 5 + j6$, $Z_2 = 6 - j7$, $Z_3 = 7 + j8$, $Z_4 = 8 - j9$, $Z_5 = 9 + j10$, $Z_6 = 10 - j11$, $Z_7 = 11 - j12$, $Z_8 = 12 + j13$; and let the impressed voltage be $80 + j90$. Determine the value of the branch currents.

CHAPTER XI

COUPLED CIRCUITS

1. General Considerations. — Heretofore we have dealt with circuits having self-contained characteristics, namely, R , $\mathcal{L}$, and C . It has been assumed that such circuits are far enough removed from other circuits that they are not affected by any electrical changes occurring in the latter. Such is not always the case. The transformer is an illustration of two electrical circuits (the primary and secondary windings) which are so close to each other that whatever electrical changes occur in one are faithfully repeated in the other. Again, variations in the armature current of an electrical machine affect the field current on account of the proximity of the two windings. Moreover, in the armature winding of a continuous current machine, a change in the current of the coil undergoing commutation has an effect on the current of the coil placed in the same slot. Finally, the operation of some radio circuits is dependent upon the effect produced by the proximity of one circuit to another. This chapter is devoted to a study of the phenomena occurring in some of these circuits.

2. Coupled Circuits. — Electric circuits which are so located or interconnected as to allow a transfer of energy from one to the other are termed coupled circuits.

The following types of coupling may occur:

- (a) *Resistance Coupling.* In this type of coupling (Fig. 1a) two or more circuits are electrically connected through a common resistance R_{12} .
- (b) *Capacitance Coupling.* The common connection between the two circuits is, in this case, a capacitor C_{12} (Fig. 1b).
- (c) *Inductance Coupling (Direct Coupling).* The common connection between the coupled circuits is, in this type, the inductance $\mathcal{L}_{12}$ (Fig. 1c).
- (d) *Mutual Inductance Coupling (Inductive Coupling).* In this type of coupling, the coupled circuits are not electrically connected at all. The coupling (see Fig. 1d) is effected through the mutual inductance $\mathfrak{M}$ of the two circuits.
- (e) *Combination Coupling.* Here the common connection consists of two or more circuit characteristics in series or in parallel as shown in Figs. 1e and 1f.

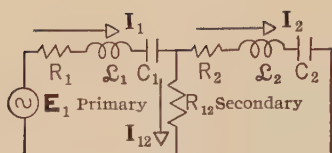


FIG. 1a. Resistance coupling.

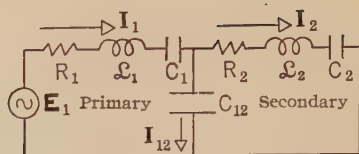


FIG. 1b. Capacitance coupling.

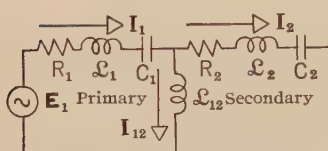


FIG. 1c. Inductance coupling.

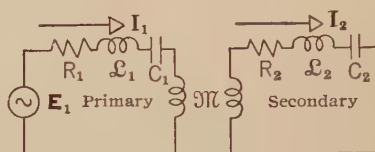


FIG. 1d. Mutual inductance coupling.

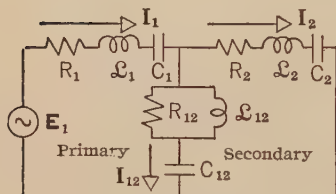


FIG. 1e. Combination coupling.

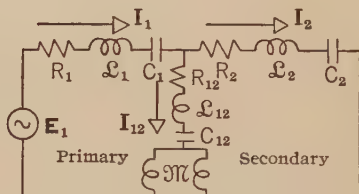


FIG. 1f. Combination coupling.

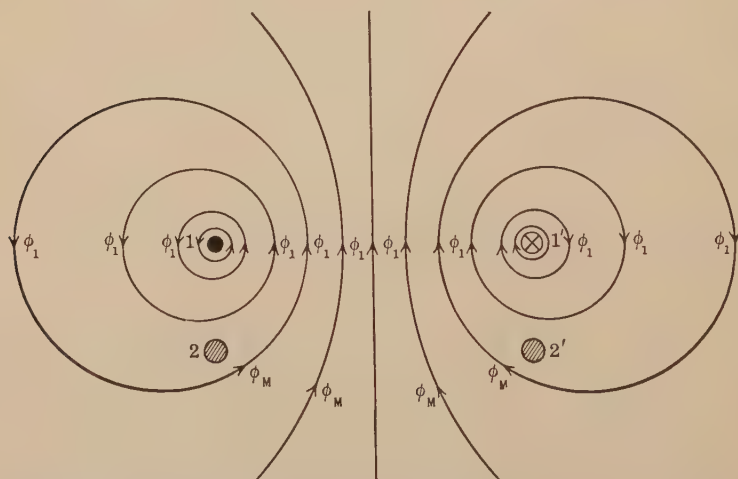


FIG. 2. A circuit 11' carrying current and the flux linking with it and with an adjacent circuit 22'.

The circuits shown in Figs. 1*a*, 1*b*, 1*c*, and 1*e* are treated in Chapter X. We shall here discuss the circuit shown in Fig. 1*d*.

3. The Concept of Mutual Inductance. — Consider the cross-sections of two coils 11' and 22' (Fig. 2). Let the coil 11' be connected to a continuous Emf. source such that in section 1 the current flows toward the reader while in section 1' the current flows away from the reader. The directions of current flow are indicated by a dot and a cross as explained in Art. 3*a*, Chapter V. The lines of flux surrounding the current are shown as circles which not only link the coil 1 (hereafter called the primary) but also (in part) link coil 2 (the secondary). Furthermore, as the secondary is brought closer to the primary more lines of flux link the former.

Now let the terminals of the secondary be kept open and let the current flowing through the primary be varied. The variation in i_1 is accompanied by a variation in the flux linking both coils and consequently induces a voltage in both. This voltage is such as to oppose the change of flux (see Art. 3*a*, Chapter V). If the total flux is ϕ and that part of it which links both coils is ϕ_{M1} , then the voltages induced in the primary and secondary will be respectively:

$$\left. \begin{aligned} e_{1L} &= -N_1(d\phi_1/dt) & (a) \\ e_{2M} &= -N_2(d\phi_{M1}/dt) & (b) \end{aligned} \right\} \quad (1)$$

The reader's attention is called to the following facts in connection with Eqs. (1):

- (1) ϕ_1 and ϕ_{M1} are produced by one and the same current i_1 .
- (2) Since ϕ_1 is the total flux produced by i_1 and ϕ_{M1} is only that part of ϕ_1 which links both the primary and secondary (see Fig. 2), ϕ_{M1} is never greater than ϕ_1 . ϕ_{M1} is known as the mutual flux of the primary because it is common to both coils.
- (3) e_{1L} is the Emf. of self-induction of circuit 1. It is induced by a change in the total flux ϕ_1 due to its own (the primary) current.
- (4) e_{2M} is the Emf. of mutual induction of the primary on the secondary. This Emf. is caused by a change in ϕ_{M1} — a flux due entirely to the primary. In general the Emf. of mutual induction is an Emf. induced in one circuit through a change in the magnetic field due to another circuit.

If the material of both circuits is of constant magnetic permeability and if the permeability of the ambient medium is also constant, then the fluxes ϕ_1 and ϕ_{M1} would be proportional to i_1 , or:

$$\left. \begin{aligned} \phi_1 &= (\mathfrak{L}_1/N_1)i_1 & (a) \\ \phi_{M1} &= (\mathfrak{M}_{12}/N_2)i_1 & (b) \end{aligned} \right\} \quad (2)$$

Substituting these values in Eqs. (1) we have:

$$\left. \begin{aligned} e_{1L} &= -\mathfrak{L}_1(di_1/dt) & (a) \\ e_{2M} &= -\mathfrak{M}_{12}(di_1/dt) & (b) \end{aligned} \right\} \quad (3)$$

where $\mathfrak{L}_1$ is the self-inductance of the primary and $\mathfrak{M}_{12}$ is the mutual inductance of the primary on the secondary. $\mathfrak{M}_{12}$ may thus be defined as the coefficient of proportionality between the voltage induced in the secondary and the time rate of change of the primary current.

Next consider the same experiment performed with coil 2 carrying a current i_2 and coil 1 open. Reasoning as above we have:

$$\left. \begin{aligned} e_{2L} &= -N_2 d\phi_2/dt = -\mathfrak{L}_2 di_2/dt & (a) \\ e_{1M} &= -N_1 d\phi_{M2}/dt = -\mathfrak{M}_{21} di_2/dt & (b) \end{aligned} \right\} \quad (4)$$

where as in Eqs. (2)

$$\left. \begin{aligned} \phi_2 &= (\mathfrak{L}_2/N_2)i_2 & (a) \\ \phi_{M2} &= (\mathfrak{M}_{21}/N_1)i_2 & (b) \end{aligned} \right\} \quad (5)$$

$\mathfrak{L}_2$ is the self-inductance of the secondary and $\mathfrak{M}_{21}$ is the mutual inductance of the secondary on the primary.

Finally, when both circuits are excited and the currents i_1 and i_2 are varied simultaneously there result two Emfs. of mutual induction and two Emfs. of self-induction. These four Emfs. are expressed by Eqs. (3) and (4) respectively.

4. Factors Affecting Mutual Inductance. — The mutual inductance between two inductively coupled circuits is experimentally found to be dependent upon the following factors:

- (a) The number of turns N_1 and N_2 in the primary and secondary respectively.
- (b) The distance between the two circuits.
- (c) The relative position of the two coils with respect to each other, that is, if coil 2 (Fig. 2) is turned on its axis through 90° and so placed that its center corresponds with that of the primary coil, then any change of current in either circuit will produce no effect on the other. This is true because the flux due to one coil does not link the other when the coils assume this position. Again, if coil 2 is further turned on its axis such that the sides 2 and 2' are interchanged, then the effect observed by a change of current will always be contrary to that observed when 2 and 2' are as indicated in Fig. 2. From this we conclude that the mutual inductance $\mathfrak{M}$ may be positive, negative, or zero. $\mathfrak{M}$ is positive if the induced voltage e_M is such as to oppose the impressed voltage.

- (d) The relative dimensions of the primary and secondary coils, that is, their relative diameters, axial lengths, thickness of windings, etc.

5. Mutual Inductance Expressed in Terms of Flux Linkages. — Solving Eqs. (2) for $\mathcal{L}_1$ and $\mathfrak{M}_{12}$ and Eqs. (5) for $\mathcal{L}_2$ and $\mathfrak{M}_{21}$:

$$\left. \begin{array}{ll} \mathcal{L}_1 = N_1 \phi_1 / i_1 & (a) \\ \mathfrak{M}_{12} = N_2 \phi_{M1} / i_1 & (b) \end{array} \right\} \quad \left. \begin{array}{ll} \mathcal{L}_2 = N_2 \phi_2 / i_2 & (c) \\ \mathfrak{M}_{21} = N_1 \phi_{M2} / i_2 & (d) \end{array} \right\} \quad (6)$$

Eq. (6a) defines the self-inductance of the primary as the total flux per unit current of the primary which links with the primary turns.

Eq. (6b) defines the mutual inductance of the primary on the secondary as the amount of mutual flux, per unit current of the primary, which links with the turns of the secondary.

Similar definitions of the self-inductance $\mathcal{L}_2$ of the secondary and of the mutual inductance $\mathfrak{M}_{21}$ of the secondary on the primary may be deduced from Eqs. (6c) and (6d) respectively.

6. Proof that $\mathfrak{M}_{12}$ and $\mathfrak{M}_{21}$ in Eqs. (6b) and (6d) are Equal. — Consider the coupled circuits shown in Fig. 3. Let the primary circuit be open while the secondary is closed and carries a constant current I_2 .

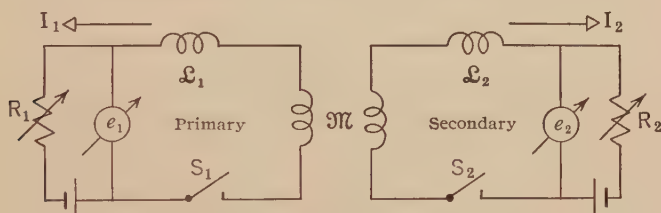


FIG. 3. Two mutually coupled circuits.

Now close the switch S_1 and vary the voltage e_2 across the secondary so as to maintain the current I_2 constant throughout the operation. The induced voltage in the secondary due to a change in I_1 is expressed by Eq. (3b). Moreover, e_{2M} is actually the value by which the voltage e_2 has to be varied in order to keep I_2 constant. Therefore, the energy supplied or received by the secondary is:

$$W_{2M} = \int_0^{I_1'} I_2 e_{2M} dt = \int_0^{I_1'} I_2 \mathfrak{M}_{12} (di_1 / dt) dt = \int_0^{I_1'} I_2 \mathfrak{M}_{12} di_1 = \mathfrak{M}_{12} I_2 I_1 \quad (7)$$

Next assume the operation outlined above to have been reversed, that is, consider the secondary circuit to have been open, and the primary circuit to have had a current I_1 . Let the switch S_2 be closed and let the secondary current increase from 0 to I_2 (keeping I_1 constant). The

voltage induced in the primary due to a change in the secondary current is expressed by Eq. (4b). Again the total energy supplied or received by the primary, due to mutual inductance, is:

$$W_{1M} = \int_0^t I_1 e_{1M} dt = \int_0^t I_1 \mathfrak{M}_{21} (di_2/dt) dt = \int_0^{I_2} \mathfrak{M}_{21} I_1 di_2 = \mathfrak{M}_{21} I_1 I_2 \quad (8)$$

Now under established conditions the total electromagnetic energy W stored in the circuits, due to both their self- and mutual-inductance, is:

$$\text{or } \quad \left. \begin{aligned} W &= 0.5 \mathfrak{L}_1 I_1^2 + \mathfrak{M}_{12} I_1 I_2 + 0.5 \mathfrak{L}_2 I_2^2 & (a) \\ W &= 0.5 \mathfrak{L}_1 I_1^2 + \mathfrak{M}_{21} I_1 I_2 + 0.5 \mathfrak{L}_2 I_2^2 & (b) \end{aligned} \right\} \quad (9)$$

But this total stored energy is independent of the order in which the switches are operated. Eqs. (9a) and (9b) are therefore equal and we have:

$$\mathfrak{M}_{12} = \mathfrak{M}_{21} = \mathfrak{M} \quad (10)$$

which means that there exists but one value of the mutual inductance $\mathfrak{M}$ between two circuits. In other words the mutual flux, per unit current of the primary, which links with the secondary is equal to the mutual flux, per unit current of the secondary, which links with the primary.

7. Relation Between Mutual Inductance and Self Inductance; The Coupling Coefficient. — The similarity in the physical phenomena underlying the concepts of self-inductance and mutual inductance suggests some relation which exists between the two quantities. Thus multiplying Eq. (6a) by (6c) and (6b) by (6d) and dividing the products we have:

$$k^2 = (\mathfrak{M}^2 / \mathfrak{L}_1 \mathfrak{L}_2) = (\phi_{M1} \phi_{M2} / \phi_1 \phi_2) \quad (11)$$

The ratio k is known as the *coupling coefficient*. Since $\phi_{M1} \leq \phi_1$ and $\phi_{M2} \leq \phi_2$ it follows that k can never be greater than unity.

The coupling coefficient is a measure of the extent to which the value of the mutual flux approaches that of the total flux or of the extent of "coupling." Two circuits are said to be loosely coupled when k is small or when the mutual fluxes are small as compared with the total fluxes. The value of k approaches unity when $\phi_{M1} \rightarrow \phi_1$ and $\phi_{M2} \rightarrow \phi_2$. Physically k can never be unity irrespective of how closely the primary and secondary are brought together. An approach to unity is, however, possible if the coils are made of very thin wire and are closely wound together.

8. Current and Voltage Relations in Coupled Circuits Subjected to Sine Emfs. — Consider the coupled circuits (Fig. 1d). The following Kirchhoff's Emf. equations hold true for the primary and secondary

respectively :

$$\left. \begin{aligned} e_1 &= R i_1 + \mathfrak{L}_1(di_1/dt) + \left(\int i_1 dt / C_1 \right) + \mathfrak{M} di_2/dt \quad (a) \\ 0 &= R i_2 + \mathfrak{L}_2(di_2/dt) + \left(\int i_2 dt / C_2 \right) + \mathfrak{M} di_1/dt \quad (b) \end{aligned} \right\} \quad (12)$$

Eqs. (12) are the most general expressions for the relation between the voltage and current in an inductively coupled circuit. They hold true irrespective of the mode of variation of e or i with t .

Consider now the special case where e_1 , i_1 , and i_2 are all sine functions of time; that is, let:

$$\left. \begin{aligned} e_1 &= E_m \sin(\omega t + \alpha) \quad (a) \\ i_1 &= I_{m1} \sin(\omega t + \beta) \quad (b) \\ i_2 &= I_{m2} \sin(\omega t + \theta) \quad (c) \end{aligned} \right\} \quad (13)$$

Substituting Eqs. (13) in (12) we have:

$$\left. \begin{aligned} E_m \sin(\omega t + \alpha) &= R I_{m1} \sin(\omega t + \beta) + \mathfrak{L}_1 \omega I_{m1} \sin(\omega t + \beta + \pi/2) \\ &\quad + (I_{m1}/C_1 \omega) \sin(\omega t + \beta - \pi/2) + \mathfrak{M} \omega I_{m2} \sin(\omega t + \theta + \pi/2) \quad (a) \\ 0 &= R I_{m2} \sin(\omega t + \theta) + \mathfrak{L}_2 \omega I_{m2} \sin(\omega t + \theta + \pi/2) + \\ &\quad (I_{m2}/C_2 \omega) \sin(\omega t + \theta - \pi/2) + \mathfrak{M} \omega I_{m1} \sin(\omega t + \beta + \pi/2) \quad (b) \end{aligned} \right\} \quad (14)$$

Casting Eqs. (14) into the Exponential form we have:

$$\left. \begin{aligned} E_1 \epsilon^{j\alpha} &= R I_1 \epsilon^{j\beta} + j(X_{L1} - X_{C1}) I_1 \epsilon^{j\beta} + jX_M I_2 \epsilon^{j\theta} \quad (a) \\ 0 &= R I_2 \epsilon^{j\theta} + j(X_{L2} - X_{C2}) I_2 \epsilon^{j\theta} + jX_M I_1 \epsilon^{j\beta} \quad (b) \end{aligned} \right\} \quad (15)$$

$$\text{Where} \quad X_M = 2 \pi f \mathfrak{M} = \omega \mathfrak{M} \quad (16)$$

X_M is known as the mutual reactance of the coupled circuits. Changing Eqs. (15) into the vectorial form we obtain:

$$\left. \begin{aligned} \mathbf{E}_1 &= Z_1 \mathbf{I}_1 + jX_M \mathbf{I}_2 \quad (a) \\ 0 &= Z_2 \mathbf{I}_2 + jX_M \mathbf{I}_1 \quad (b) \end{aligned} \right\} \quad (17)$$

Eqs. (16) are the most general vectorial expressions for the steady-state in an inductively coupled circuit whose primary is subjected to a sine voltage.

9. The Concept of Equivalence in Inductively Coupled Circuits. — Consider the coupled circuits shown in Fig. 1*d*. Let a three-element oscillograph be properly connected in the circuits to give e_1 , i_1 , and i_2 . Can these circuits be represented by equivalent simple series circuits and what is meant by equivalence when more than one current is involved?

We shall define two equivalent circuits the one referred to the primary and the other referred to the secondary.

(a) The equivalent circuit referred to the primary is such a simple series circuit that, when a vector voltage $\mathbf{E}_1$ is impressed on it, it gives a vector current $\mathbf{I}_1$.

(b) The equivalent circuit referred to the secondary is such a simple series circuit which, when subjected to the same vector voltage $\mathbf{E}_1$, gives a vector current $\mathbf{I}_2$.

Relations will now be deduced which give the characteristics of these pure series circuits in terms of the characteristics of the coupled circuits.

(a) *The Equivalent Impedance Referred to the Primary.* — Multiply Eq. (17a) by Z_2 and (17b) by jX_M and subtract thereby obtaining:

$$\mathbf{I}_1 = \mathbf{E}_1/[Z_1 + (X_M^2/Z_2)] = \mathbf{E}_1/Z' \quad (18)$$

Thus the impedance of the equivalent circuit referred to the primary is:

$$Z' = Z_1 + (X_M^2/Z_2) \quad (19)$$

(b) *The Equivalent Impedance Referred to the Secondary.* — Eliminating $\mathbf{I}_1$ from Eqs. (17) and solving for $\mathbf{I}_2$ we obtain:

$$\mathbf{I}_2 = -jX_M\mathbf{E}_1/(Z_1Z_2 + X_M^2) = \mathbf{E}_1/Z'' \quad (20)$$

Hence the impedance of the equivalent circuit referred to the secondary is:

$$Z'' = -jX_M/(Z_1Z_2 + X_M^2) \quad (21)$$

Physically this means that if two circuits of impedances Z' and Z'' are simultaneously connected to a voltage source $\mathbf{E}_1$, then the oscillogram of e , i_1 , and i_2 will be identical with that obtained for the coupled circuits (Fig. 1d) when an Emf. $\mathbf{E}_1$ is impressed on the primary. Evidently then coupled circuits, although falling in a separate class and forming a distinct type, present nothing new and lend themselves to a transformation which enables one to treat them as pure series circuits. The primary and secondary currents in coupled circuits may be computed by the use of Eqs. (18) and (20) respectively.

10. Partial Resonance in Coupled Circuits. — Partial resonance in coupled circuits is obtained when any one characteristic is so varied as to cause the secondary current to attain a maximum effective value when a constant Emf. is impressed on the primary circuit.

By definition, the effective value of I_2 must be a maximum. This effective value may be obtained by rationalizing Eq. (20). The process consists of expressing Z_1 and Z_2 in the orthogonal form, expanding the product then multiplying the numerator and denominator by the con-

jugate of the denominator and simplifying. This gives:

$$I_2 = X_M E_1 / [Z_1^2 Z_2^2 + 2 X_M^2 (R_1 R_2 - X_1 X_2) + X_M^4]^{1/2} \quad (22)$$

Eq. (22) shows that I_2 is a function of all the characteristics of the circuits.

Coupled circuits may therefore have at least five resonant points obtained through a variation of R_1 , X_1 , R_2 , X_2 , or X_M , respectively.

We shall derive the partial resonance relation when one of these characteristics is varied, leaving similar derivations for the remaining characteristics as exercises to the student.

Consider the circuit Fig. 1d. Let E , R_1 , R_2 , X_2 , and X_M be held constant and let x_1 be varied until the maximum value of I_2 (resonance) is attained. It is required to determine the value of x_1 which gives resonance.

Solution. — Since X_M and E_1 are constant, the effective value of I_2 will be a maximum when the denominator (or the square of the denominator) in Eq. (22) is a minimum. In order to find the value of x_1 which renders the square of the denominator a minimum it is necessary to differentiate with respect to x_1 , equate the derivative to zero, and solve for x_1 . Thus squaring the denominator in Eq. (22) and replacing Z_1^2 by $R_1^2 + x_1^2$ we have:

$$D = (R_1^2 + x_1^2)Z_2^2 + 2 X_M^2 (R_1 R_2 - x_1 X_2) + X_M^4 \quad (23)$$

Differentiating Eq. (23) with respect to x_1 equating the derivative to zero and solving for x_1 we have:

$$x_1 = X_M^2 X_2 / Z_2^2 \quad (24)$$

Thus maximum secondary current is obtained in a coupled circuit when its primary reactance is so adjusted as to satisfy Eq. (24). A question arises as to whether x_1 is inductive or capacitive. The answer is that it will have to be the same as X_2 . If X_2 is inductive then x_1 is also inductive.

PROBLEMS

1. Cite instances, other than those given in Art. 1, where mutual inductance occurs in engineering practice.

2. Draw figures of combination coupling other than those shown in Figs. 1e and 1f.

3a. A coil (b) having 100 turns of thin copper wire is placed next to another coil (a) carrying a sine current. Assuming that the mutual flux linked with both coils has an amplitude of 10^{-7} webers, what would be the instantaneous value of the voltage induced in the coil (b) if the frequency of the source is 3×10^6 cycles per second?

3b. The amplitude of the primary current in problem 3a is 10^{-2} amp. Compute the mutual inductance of the coupled coils.

3c. Two coils are placed next to each other and a battery is connected to one of them (the primary) through a variable resistance r . When the current in the primary is varied at the rate of 500 amp. per second, the voltage induced in the secondary is found to be 0.2 volt. Determine the mutual inductance of the coupled circuits.

3d. Two circuits are coupled such that the mutual inductance of the primary on the secondary is 0.05 mh. Assuming that the primary current is 10^{-3} amp. let the secondary turns be 15. Compute the mutual flux.

3e. Let the mutual flux in problem 3d be 5 per cent of the primary flux and let the turns of the primary be 8. Compute the primary inductance.

3f. Two circuits are so coupled together that the ratio of the mutual flux of the primary to its total flux is 25 per cent. If the primary and secondary turns are equal what is the ratio of the self-inductance of the primary to the mutual inductance of the coupled circuits.

3g. The primary current in the circuit of problem 3f is varied, while the secondary is kept open. The Emf. induced in the primary due to its current variation is 0.5 volt. What is the Emf. induced in the secondary?

4. Show how you would wind the primary and secondary of an inductively coupled circuit such that the mutual inductance is positive.

5a. The ratio of the mutual to the total flux of the primary is 0.2. If the turns of the primary and secondary are such that $(N_1/N_2) = 10$. Determine the value of $\mathfrak{M}_{12}$ in terms of $\mathfrak{L}_1$.

5b. The flux per unit current of the primary is 10^{-7} webers. What is the inductance of the primary per turn?

5c. If the flux which links the secondary per unit current of the primary is 10^{-8} webers, determine the mutual inductance per turn of the secondary.

6a. The continuous current in the primary of an inductively coupled circuit (Fig. 3) is 10 amp., while that in the secondary is 5 amp. Moreover, the mutual inductance is 10^{-4} henry. What is the energy stored due to mutual inductance?

6b. Assume that the self-inductances of the primary and secondary in problem 6a are $\mathfrak{L}_1 = 10^{-3}$ and $\mathfrak{L}_2 = 10^{-4}$ henry. Determine the total electromagnetic energy stored in the coupled circuits when the established currents in the primary and secondary are 20 and 15 amp. respectively.

6c. Devise some means of establishing the existence of stored energy in the circuit, Fig. 3.

6d. Assume the coupled circuits, Fig. 3, to be carrying currents as specified in problem 6b. Now let the circuits be drawn apart thereby decreasing their mutual inductance. Outline in detail just what would happen in each circuit.

7. The inductances of the primary and secondary of two coupled circuits are $\mathfrak{L}_1 = 5$ mh.; $\mathfrak{L}_2 = 2$ mh. If the mutual inductance is 3 mh., what is the coefficient of coupling?

8a. Show that in two inductively coupled circuits the coupling coefficient $k = X_M / \sqrt{X_{L1} X_{L2}}$.

8b. Determine the mutual inductive reactance X_M of the coupled circuits specified in problem 3b if $f = 10,000$ cycles per second.

8c. The resistance and capacitance of the primary and secondary of two coupled circuits are negligible. Show that the primary current $I_1 = E_1/jX_1(1 - k^2)$.

8d. One way of measuring the mutual inductance of two coupled circuits consists of (a) determining the characteristics Z_2 of the secondary by the ordinary bridge methods, and then (b) constructing a bridge circuit similar to that of Fig. 4 and balancing the circuit until the galvanometer (or detector) current is zero. The vector equations are (see Fig. 4):

$$\begin{aligned} I_1 &= I_2 + I_3 \\ I_3 Z_b &= I_2 Z_c \\ I_3 Z_a &= I_2 Z_2 + jX_M I_1 \end{aligned}$$

(a) Explain how these equations are obtained.

(b) Express X_M in terms of the known characteristics Z_a , Z_b , Z_2 , and Z_d

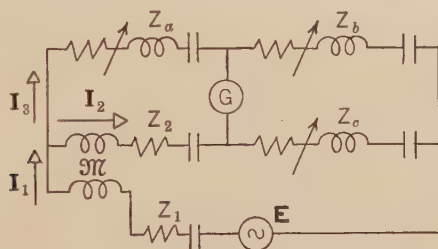


FIG. 4. A bridge circuit for measuring mutual inductance.

9a. Two inductively coupled circuits have the following characteristics:

$$\begin{array}{lll} R_1 = 3 \text{ ohms} & R_2 = 40 \text{ ohms} & k = 0.72 \\ \mathcal{L}_1 = 0.05 \text{ mh.} & \mathcal{L}_2 = 0.02 \text{ mh.} & \\ C_1 = 0.07 \mu \text{ f.} & C_2 = 0.06 \mu \text{ f.} & \end{array}$$

If a 60,000-cycle 1-volt wave is impressed on the primary, what would be the complex expressions for I_1 and I_2 ?

Note. — Consider $E = 0 + j1$.

9b. An air-core transformer is designed to operate at 60 cycles and 110 primary volts. The characteristics of its circuits are:

$$\begin{array}{lll} R_1 = 10 \text{ ohms} & R_2 = 2 \text{ ohms} & X_M = 0.5 \text{ ohm} \\ X_{L1} = 5 \text{ ohms} & X_{L2} = 1 \text{ ohm} & \end{array}$$

Assuming that the impressed voltage is $0 + j110$:

(a) Compute the primary and secondary currents.

(b) Draw a vector diagram showing:

- (1) impressed voltage.
- (2) voltage drops in the primary.
- (3) voltage drops in the secondary.

(c) Using the vector diagram, explain the operation of this apparatus.

9c. An instrument transformer consists of a primary and secondary winding placed on an iron core. For purposes of accuracy the core is so designed that its permeability is practically constant. Let the characteristics of a potential transformer be:

$$\begin{array}{lll} R_1 = 50 \text{ ohms} & R_2 = 5 \text{ ohms} & X_M = 25 \text{ ohms} \\ X_{L1} = 80 \text{ ohms} & X_{L2} = 8 \text{ ohms} & \end{array}$$

Assuming that the primary voltage is $650 + j900$

(a) Compute the primary and secondary currents.

(b) Draw a completely labeled vector diagram of the transformer.

9d. Show that the effect of the secondary of a transformer is to increase its primary resistance by an amount equal to $X_M^2 R_2 / Z_2^2$ and to reduce its primary reactance by an amount equal to $X_M^2 X_2 / Z_2^2$.

9e. Two inductively coupled circuits have the following characteristics:

$$\begin{array}{lll} R_1 = 5 \text{ ohms} & \mathcal{L}_1 = 0.3 \text{ mh.} & \mathfrak{M} = 0.15 \text{ mh.} \\ R_2 = 4 \text{ ohms} & \mathcal{L}_2 = 0.2 \text{ mh.} & \end{array}$$

If the frequency of the voltage wave impressed on the primary is:

$$3 \times 10^3 \text{ cycles per second,}$$

what is the impedance of the equivalent series circuit referred to the primary?

9f. Determine the impedance of the series circuit, referred to the primary, which is equivalent to the coupled circuits specified in problem 9e if the voltage wave is applied to the secondary instead of to the primary. Compare this impedance with the impedance obtained in problem 9e.

10a. Supply the missing steps between Eqs. (20) and (22).

10b. Prove that partial resonance in two inductively coupled circuits may be had at a value of $X_2 = X_1 X_M^2 / Z_1^2$.

10c. Show that partial resonance may be obtained if the primary resistance satisfied the following conditions:

$$(a) R_1 = 0$$

$$(b) R_1 = -R_2 X_M^2 / Z_2^2$$

Is the second condition possible? Explain.

10d. Give the conditions that the secondary resistance must satisfy in order to obtain partial resonance. Discuss your results.

CHAPTER XII

ENERGY AND POWER IN ALTERNATING ELECTRIC CIRCUITS

1. Electric Energy. — The electric energy delivered (or work done) by an Emf. source E , which supplies a current I for a time t , is defined in Chapter V through the relation:

$$W = EIt \quad (1)$$

Eq. (1) holds true when E and I are constant and independent of time. If E and I are functions of time, then the increment of energy dW supplied in time dt is:

$$dW = ei \, dt \quad (2)$$

and the total energy supplied in the interval of time $t = t_2 - t_1$ is:

$$W = \int_{t_1}^{t_2} ei \, dt \quad (3)$$

2. Electric Power. — In some electrical problems it is desired to know the rate at which energy is delivered, rather than the total energy supplied. This rate is termed power. By definition:

$$p = (dW/dt) = ei \quad (4)$$

Distinction will be drawn between (a) Instantaneous power and (b) Average power.

(a) *The Instantaneous Power* p is the rate at which energy is delivered at every instant. It is simply the instantaneous value of the product ei . This value is expressed by Eq. (4).

(b) *The Average Power (or Power)* P is the average rate at which energy is supplied over a certain prescribed period of time $t = t_2 - t_1$. The process of determining the average power consists in computing the total energy W supplied in a period of time t and dividing this energy by the time. The mathematical expression for average power is:

$$P = (W/t) = (1/t) \int_{t_1}^{t_2} ei \, dt = (1/t) \int_{t_1}^{t_2} p \, dt \quad (5)$$

Indicating wattmeters are constructed to read power over a complete period of the Emf. wave. Thus in the special case where $t = T$, Eq.

(5) becomes:

$$P = (W/T) = (1/T) \int_0^T ei \, dt = (1/T) \int_0^T p \, dt \quad (6)$$

3. The General Expressions for Power in a Circuit Subjected to a Single Sine Emf. — A review of all the circuits which are excited by a single Emf. source shows that the phase angle ϕ between the impressed voltage e and the total current i varies from $-\pi/2$ to $+\pi/2$ depending upon the characteristics of the circuit. In general, therefore, the Emf. of the source and the total current it furnishes may be expressed as follows:

$$\left. \begin{aligned} e &= E_m \sin (\omega t + \alpha) & (a) \\ i &= I_m \sin (\omega t + \alpha + \phi) & (b) \end{aligned} \right\} \quad (7)$$

where $-\pi/2 \leq \phi \leq \pi/2$.

By substituting Eqs. (7) in (4) and (6) we obtain two general expressions for instantaneous and average power respectively.

(a) *The General Expression for Instantaneous Power.* — Substituting Eqs. (7) in (4) we have:

$$p = E_m I_m \sin (\omega t + \alpha) \sin (\omega t + \alpha + \phi) \quad (8)$$

Using the trigonometric transformations $\sin a \sin b = [\cos (a - b) - \cos (a + b)]/2$ we obtain:

$$p = E_m I_m / 2 [\cos \phi - \cos (2 \omega t + 2 \alpha + \phi)] \quad (9)$$

Making use of the fact that $\cos (a - b) = \cos a \cos b + \sin a \sin b$, and taking $a = 2 (\omega t + \alpha + \phi)$, and $b = \phi$, Eq. (9) becomes:

$$\left. \begin{aligned} p &= (E_m I_m / 2) [\cos \phi - \cos \phi \cos 2 (\omega t + \alpha + \phi) \\ &\quad - \sin \phi \sin 2 (\omega t + \alpha + \phi)] & (a) \\ \text{or } p &= EI \cos \phi [1 - \cos 2 (\omega t + \alpha + \phi)] \\ &\quad - EI \sin \phi \sin 2 (\omega t + \alpha + \phi) & (b) \end{aligned} \right\} \quad (10)$$

Eqs. (8), (9), and (10) are identical and each of them is a general expression for the instantaneous power p delivered to a circuit by a sine Emf.

The value of the instantaneous power as expressed by Eq. (10) throws light on some obscure phases of this quantity. It, therefore, deserves special consideration.

It will be observed (Eq. 10b) that instantaneous power consists of three terms:

- (1) *A Constant Term $EI \cos \phi$.* This term is independent of time and is a scalar quantity. It represents what is commonly known as average power.

- (2) *A Cosine Term* ($-EI \cos \phi \cos 2(\omega t + \alpha + \phi)$). This term is a cosine function of time. It has an amplitude equal to $EI \cos \phi$, an angular velocity 2ω and an initial angle $2(\alpha + \phi)$. Hence it may be represented by a revolving vector. This term accounts for the fact that the power supplied by a sine source is not steady (see curve p , Fig. 1c). The amplitude of this component is numerically equal to the value of the constant term. Therefore that part of the instantaneous power which consists of terms (1) and (2), namely, $EI \cos \phi [1 - \cos 2(\omega t + \alpha + \phi)]$, can never be negative. Physically this means that the sum of the first two terms in the instantaneous power expression represents the rate at which energy is dissipated by the circuit in the form of heat.
- (3) *A Sine Term* ($-EI \sin \phi \sin 2(\omega t + \alpha - \phi)$). This sine function has an amplitude equal to $EI \sin \phi$, an angular velocity 2ω , and an initial angle $2(\alpha - \phi)$. It may therefore be represented by a revolving vector. This term expresses the instantaneous rate at which energy is stored in or returned by the circuit. It is known as the reactive (or wattless) instantaneous power.

(b) *The General Expression for Average Power.*—Substituting Eq. (9) in (6) and integrating we have:

$$P = (E_m I_m / 2) \cos \phi = EI \cos \phi \quad (11)$$

Eq. (11) is the general expression for the average power (or power) delivered to a circuit by a sine Emf. Note that its value is represented by the constant term in Eq. (9) or Eq. (10b). This could have been predicted because the average value of a sine or cosine function over a complete cycle is zero. The term $\cos \phi$, in Eq. (11), is known as the *power factor* (abbreviated p.f.) of the circuit.

That Eq. (11) represents power dissipated as Joulean heat in a circuit may be shown as follows.

- (1) In a series circuit $\cos \phi = R/Z$ and $E = IZ$. Substituting these values in Eq. (11) and simplifying we have (for a series circuit only):

$$P = RI^2 \quad (12)$$

- (2) In a parallel circuit $\cos \phi = G/Y$ and $I = EY$. Thus, for a parallel circuit, Eq. (11) becomes:

$$P = GE^2 \quad (13)$$

Hence the average power dissipated in a series circuit is proportional to the square of the effective value of the current and the coefficient of proportionality is the resistance of the circuit. In a parallel circuit the power dissipated is proportional to the square of the effective value of the impressed Emf. and the coefficient of proportionality is the conductance G of the circuit.

The student is here warned against using Eqs. (12) and (13) interchangeably. Eq. (12) should be used only for series circuits and Eq. (13) only for parallel circuits. Eq. (11) is, however, general and may be used for either series or parallel circuits.

4. Application of the General Expressions for Power and Energy to Special Cases. — The general expressions for power and energy become more real to the reader when applied to special circuits. We shall consider the following cases:

- (1) A circuit with resistance R .
- (2) A circuit with inductance $\mathfrak{L}$.
- (3) A circuit with capacitance C .
- (4) A series circuit with $\mathfrak{L}$ and C .
- (5) A series circuit with R and $\mathfrak{L}$.

(1) **POWER AND ENERGY IN A CIRCUIT CONSISTING OF A NON-INDUCTIVE RESISTANCE.** (a) *Instantaneous Power.* — Fig. 1a shows a circuit consisting of a non-inductive resistance and a sine Emf. The voltage and current curves, which in this case would be in phase, are shown in Fig. 1b. The curve of instantaneous power p is obtained, according to Eq. (8), by multiplying corresponding values of e and i and plotting the products, to a convenient scale, as ordinates.

The voltage and current are in phase. Hence $\phi = 0$, $\cos \phi = 1$ and $\sin \phi = 0$. Substituting these values in Eq. (10b) we have:

$$p = EI [1 - \cos 2(\omega t + \alpha)] \quad (14)$$

which shows that the instantaneous power curve may be considered as consisting of a constant term (EI) and a double frequency cosine term $EI \cos 2(\omega t + \alpha)$. These two components of the power curve are shown graphically in Fig. 1c as P_1 and p_2 respectively. Their sum is p .

In Eq. (14) $\cos 2(\omega t + \alpha)$ varies from -1 to $+1$. Substituting these two limits we have:

$$\left. \begin{aligned} p_{\max.} &= 2EI = E_m I_m & (a) \\ p_{\min.} &= 0 & (b) \end{aligned} \right\} \quad (15)$$

Thus, in a resistance circuit, the instantaneous power varies from 0 to $E_m I_m$ (see Fig. 1c). It is never negative. Physically this means

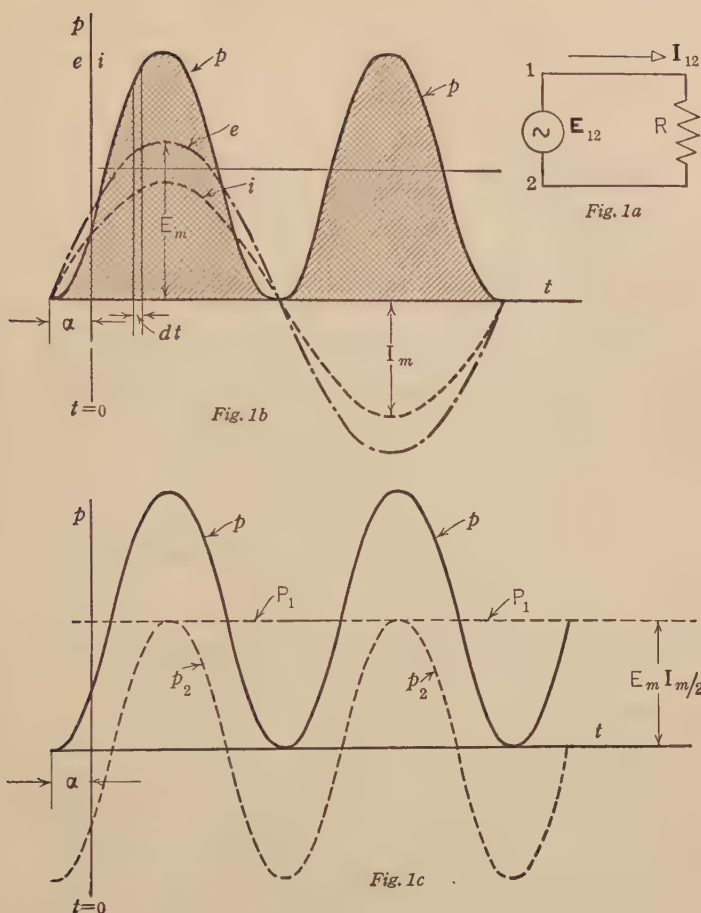


FIG. 1a. A series circuit consisting of a resistance and a sine Emf.

FIG. 1b. Emf., current, power, and energy in the circuit Fig. 1a.

FIG. 1c. The power curve p subdivided into a constant component P_1 and a double frequency sine component p_2 .

that whatever power is supplied to the circuit is consumed by the resistance and appears as heat. None of this power ever returns to the Emf. source.

Note that in this circuit only the first two terms, of the instantaneous power expression (Eq. 10), are present. Since energy can be stored only when a circuit possesses $\mathcal{L}$ or C , there can be no energy stored in the circuit Fig. 1a. This is a further justification for the assertion that the sum of the first two terms in the power expression represents dissipated power.

(b) *Average Power.* — Substituting $\cos \phi = 1$ in Eq. (11) we have:

$$P = EI = (IR)I = I^2R \quad (16)$$

(c) *Energy.* — From Eq. (3) the energy dissipated in the interval 0 to t is

$$W = \int_0^t ei \, dt = \int_0^t p \, dt = Pt = EIt = I^2Rt \quad (17)$$

The energy dissipated per cycle T is represented by the shaded area Fig. 1b.

(2) POWER AND ENERGY IN AN INDUCTANCE CIRCUIT. (a) *Instantaneous Power.* — Fig. 2a shows a circuit with an inductive reactance X_L , and a sine voltage e . The current and voltage curves are shown in Fig. 2b. In this case, the current lags the voltage by 90° . The curve p , representing instantaneous power, is obtained by multi-

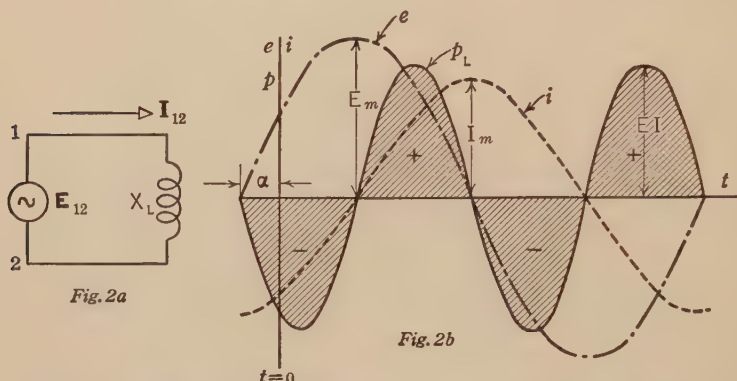


FIG. 2a. A series circuit consisting of an inductance and a sine Emf.

FIG. 2b. Emf., current, power, and energy in the circuit Fig. 2a.

plying corresponding instantaneous values e and i and plotting the products as ordinates. Note that the p -curve is a double-frequency sine curve and that it assumes positive and negative amplitudes in the same manner as the voltage or current curves. It is therefore a vector quantity.

These facts can be mathematically demonstrated as follows:

Substituting $\phi = -90^\circ$, $\cos \phi = 0$ and $\sin \phi = -1$ in Eq. (10b) we have:

$$p = EI \sin 2(\omega t + \alpha - 90^\circ) \quad (18)$$

which is a sine wave of amplitude EI , angular velocity 2ω , and initial angle $2(\alpha - 90^\circ)$.

(b) *Average Power.* — Substituting $\cos \phi = 0$ in Eq. (11) we have:

$$P = 0 \quad (19)$$

(c) *Energy*. — The shaded areas in Fig. 3b represent energy. These areas have alternate and equal positive and negative values. The positive areas represent energy supplied by the source to the inductance. This energy is stored in the inductance while the current is increasing. When the current decreases energy is returned from the inductance to the Emf. source. The returned energy is represented by the negative areas. There is no exchange of energy between $\mathcal{L}$ and e when the current is zero. Thus energy oscillates between $\mathcal{L}$ and e , being alternately stored and returned. That the net energy supplied by the source to the inductance, over a complete cycle, is zero may be shown mathematically by substituting Eq. (18) in Eq. (3) and integrating, thereby obtaining:

$$W = \int_0^T p \, dt = EI \int_0^T \sin 2(\omega t + \alpha + \phi) dt = 0 \quad (20)$$

This does not mean, however, that the energy relation is always zero, for in the interval $t = t_2 - t_1$ the energy is

$$W = EI \int_{t_1}^{t_2} \sin 2(\omega t + \alpha) dt \quad (21)$$

If Eq. (21) is positive it represents stored energy; otherwise it represents returned energy.

Note that the instantaneous value of the power for this type of circuit (Eq. 18) consists of only the last term of the general instantaneous power expression (Eqs. 10). The circuit, Fig. 2a, is by assumption devoid of resistance; hence no energy can be dissipated in it. The rate at which energy is stored in the circuit is expressed by the third term of Eq. (10). This furnishes an additional justification for subdividing the general power expression into three terms, the first two of which stand for the rate at which energy is dissipated and the last for the rate at which energy is stored.

(3) POWER AND ENERGY IN A CAPACITANCE CIRCUIT. (a) *Instantaneous Power*. — Consider a circuit, Fig. 3a, consisting of a capacitive reactance X_c and a sine Emf. e . The current voltage and power curves are shown in Fig. 3b. The p -curve is a double-frequency sine curve and is therefore a vector quantity. The instantaneous value expression for curve p (Fig. 3b) can be obtained from Eq. (10b) by substituting $\phi = 90^\circ$, $\cos \phi = 0$ and $\sin \phi = 1$. Thus:

$$p = -EI \sin 2(\omega t + \alpha + 90^\circ) \quad (22)$$

(b) *Average Power*. — Substituting $\cos \phi = 0$ in Eq. (11) we find that the average power is zero. This is in accordance with physical facts. Indeed since, by assumption, the circuit, Fig. 3a, is devoid of resistance,

there can be no power dissipated in the form of heat. Hence the average power supplied over a cycle is zero.

(c) *Energy*. — The shaded areas (Fig. 3b) represent energy stored in and returned from the circuit. The positive areas stand for energy supplied by the Emf. source and stored in the capacitor in the electrostatic form. The negative areas represent energy returned by the

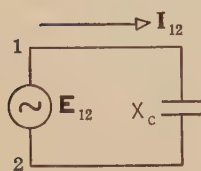


Fig. 3a

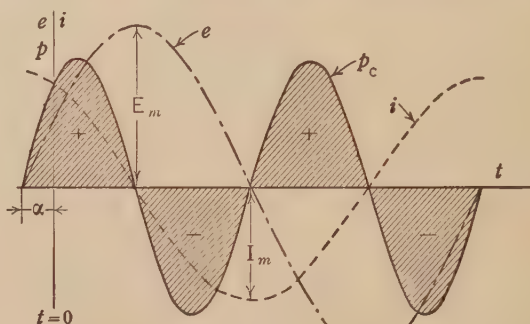


Fig. 3b

FIG. 3a. A series circuit consisting of a capacitance and a sine Emf.

FIG. 3b. Emf., current, power, and energy in the circuit Fig. 3a.

capacitor to the Emf. source. Note that the positive and negative areas are equal when taken over a complete cycle. This shows that whatever energy is stored in the capacitor during the time while the Emf. is increasing is returned while the Emf. decreases.*

(4) POWER AND ENERGY IN A SERIES CIRCUIT CONSISTING OF X_L , X_C , AND A SINE EMF. (a) *Instantaneous Power*. — The series circuit, Fig. 4a, consists of X_L , X_C and a sine Emf. E_{12} . The waves of applied Emf. potential drops, current, and total power are shown in Fig. 4b.

The three power curves (p_L , p_c , and p , Fig. 4c) represent the rate at which energy is stored or returned by the inductance, capacitance, and Emf. source respectively. These curves are obtained by multiplying the current i by e_L , e_c , and e . Thus (see Fig. 4b):

$$\left. \begin{aligned} p_L &= e_L i = X_L I_m \sin(\omega t + \alpha + \pi) I_m \sin(\omega t + \alpha + \pi/2) \\ &= -(X_L I_m^2/2) \sin 2(\omega t + \alpha) \quad (a) \\ p_c &= e_c i = X_C I_m \sin(\omega t + \alpha) I_m \sin(\omega t + \alpha + \pi/2) \\ &= (X_C I_m^2/2) \sin 2(\omega t + \alpha) \quad (b) \\ p &= e i = (X_C - X_L) I_m \sin(\omega t + \alpha) I_m \sin(\omega t + \alpha + \pi/2) \\ &= [(X_C - X_L) I_m^2/2] \sin 2(\omega t + \alpha) \quad (c) \end{aligned} \right\} \quad (23)$$

* It is here assumed that the capacitor is entirely free of any anomalous behavior, in other words that the dielectric is perfect.

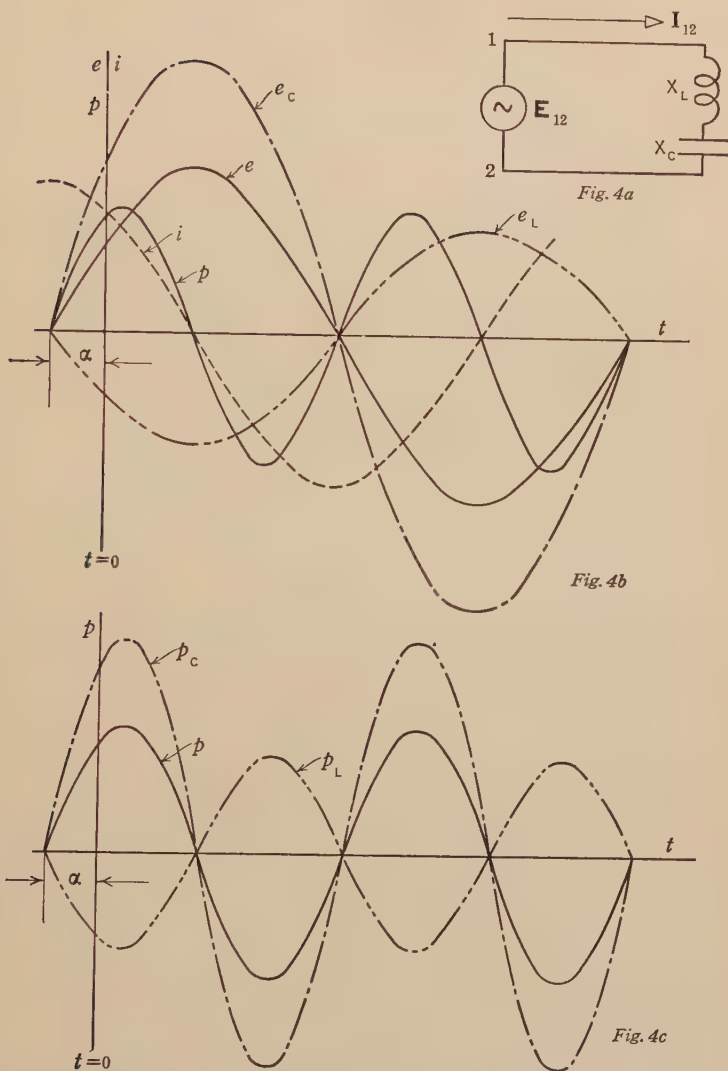


FIG. 4a. A series circuit consisting of $\mathcal{L}$, C , and E_{12} .

FIG. 4b. Sine waves of Emf., current, and potential drops in the circuit Fig. 4a.

FIG. 4c. Sine waves of power in the circuit Fig. 4a.

Note that the curves p_c and p_L (Fig. 4c) are displaced from each other by 180° . In other words, while energy is stored in the capacitor, energy is delivered by the inductance, and vice versa. There is thus an exchange of energy between the inductance and capacitance.

(b) *Average Power*. — The average power over a complete cycle supplied to the circuit, Fig. 4a, is zero because the phase displacement between e and i , e_L and i , e_c and i is $(\pi/2)$. Thus $\cos \pi/2$ is zero and P is zero.

(c) *Energy*. — The amount of energy stored or delivered by $\mathcal{L}$ or C (Fig. 4a) in a time $t = t_1 - t_2$ may be easily computed by applying Eq. (5). Fig. 4c shows, however, that the positive and negative areas under each of the curves p , p_L and p_c are equal. Consequently there exists no dissipation of energy. Whatever energy is supplied by the Emf. source is returned back with no loss whatever.

(5) POWER AND ENERGY IN A SERIES CIRCUIT WITH R AND $\mathcal{L}$.

(a) *Instantaneous Power*. — Fig. 5a shows a series circuit consisting of a resistance R , an inductance $\mathcal{L}$, and a sine Emf., $\mathbf{E}_{12}$. The current (Fig. 5b) lags the Emf. by an angle ϕ . The curve for instantaneous power has both positive and negative values. The negative areas, which represent energy returned from the circuit to the electric source, are smaller than the positive areas which represent energy supplied by the Emf. There exists, therefore, a net dissipation of energy. •

The physical phenomena occurring in this circuit may be better understood when it is remembered that the total impressed voltage may be divided into two components (see Fig. 5c). The one, e_r , is in phase with i ; and the other, e_L , leads i by 90° . The first (e_r), when multiplied by the current, gives an instantaneous power curve similar to that given in Figs. 1b and 1c for a resistance circuit. The second (e_L), when multiplied by i , gives the power curve p_L shown in Fig. 2b. The sum of these curves is the instantaneous power curve p (Figs. 5b and 5d).

These facts can be mathematically deduced from the general expression for instantaneous power. Thus making the substitution $\cos \phi = R/Z$ and $\sin \phi = X/Z$ Eq. (10b) becomes:

$$p = RI^2 [1 - \cos 2(\omega t + \alpha + \phi)] - (XI^2) \sin 2(\omega t + \alpha + \phi) \quad (24)^*$$

That the first term in Eq. (24) is the product $e_r i$ and the second term is the product $e_L i$ may be easily demonstrated as follows:

$$\left. \begin{aligned} e_r i &= (Ri)i = RI_m^2 \sin^2(\omega t + \alpha + \phi) \\ &= RI^2 [1 - \cos 2(\omega t + \alpha + \phi)] \\ e_L i &= (X_L i)i = X_L I_m^2 \sin(\omega t + \alpha + \phi) \sin(\omega t + \alpha + \phi + 90^\circ) \\ &= X_L I^2 \sin 2(\omega t + \alpha + \phi) \end{aligned} \right\} \quad (25)^*$$

* The phase angle ϕ in Eqs. (24) and (25) has a value between 0 and $-\pi/2$.

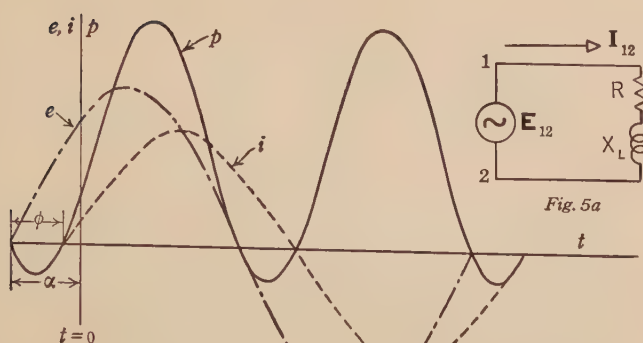


Fig. 5a



Fig. 5b

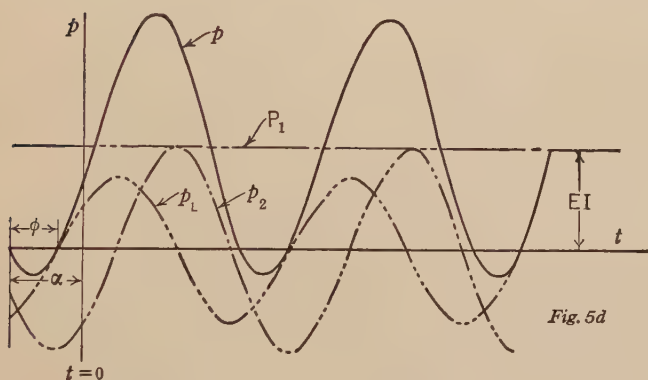


Fig. 5c

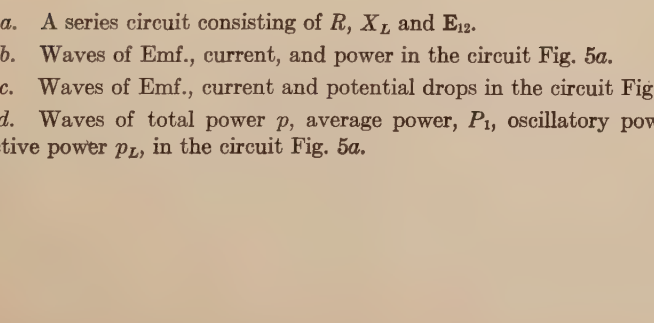


Fig. 5d

FIG. 5a. A series circuit consisting of R , X_L and E_{12} .

FIG. 5b. Waves of Emf., current, and power in the circuit Fig. 5a.

FIG. 5c. Waves of Emf., current and potential drops in the circuit Fig. 5a.

FIG. 5d. Waves of total power p , average power, P_1 , oscillatory power p_2 and reactive power p_L , in the circuit Fig. 5a.

(b) *Average Power*. — The average power in the circuit, Fig. 5a, is computed by the use of either Eq. (11) or Eq. (12).

(c) *Energy*. — Part of the energy supplied by the source (Fig. 5a) is dissipated in the resistance, while the other part is stored in the inductance. The part dissipated in the resistance in the time $t = t_2 - t_1$ is:

$$W_r = Pt \quad (26)$$

and the part stored in the inductance in the same time interval is:

$$W_L = \int_{t_1}^{t_2} p_L dt = \int_{t_1}^{t_2} EI \sin 2(\omega t + \alpha + \phi) dt \quad (27)$$

The total energy is:

$$W = W_r + W_L \quad (28)$$

where W_L may be positive or negative or zero depending on the limits taken. Positive W_L represents stored energy and negative W_L indicates returned energy.

This concludes the special cases which we have undertaken to discuss in the text. It is hoped that the student has grasped the meaning of the general expressions for power and energy and that he can apply them to other cases which are given as problems.

5. Power in Series-Parallel Circuits. — Three problems arise in computing the power supplied to a series parallel circuit such as that shown in Fig. 2, Chapter X.

- (a) To find the total power dissipated in the whole circuit.
- (b) To find the power dissipated in each branch.
- (c) To find the total power as well as the power in each branch.

(a) To find the total power dissipated, first convert the circuit to an equivalent series circuit. Then apply Eqs. (11) or (12).

(b) To find the power dissipated in each branch, first compute the various branch currents. Then the power dissipated in a branch whose terminal points are pq is by Eq. (12):

$$P_{pq} = R_{pq} I_{pq}^2 \quad (29)$$

(c) Should the total power as well as the power in each branch be required, then compute the power in each branch as outlined in part (b) and add to obtain the total power. Thus

$$P = \sum P_{pq} \quad (30)$$

6. Power in Circuits Comprising More Than One Emf. Source. — When Emf. sources and impedances are combined as in Fig. 7, Chapter

X the power supplied by the Emf. sources and dissipated in the resistances may be computed by the use of Eqs. (11), (12) or (13). According to the principle of conservation of energy, the power supplied by the Emf. sources must be equal to the power dissipated in the various resistances. Thus:

$$\sum E_{uv} I_{vu} \cos \phi = \sum I_{pq}^2 R_{pq} \quad (31)$$

where I_{vu} is the *internal current* flowing through the source E_{uv} and I_{pq} is the current traversing the resistance R_{pq} . It will be noted that each of the terms entering into the summation at the right in Eq. (31) is positive. Such is not necessarily the case with the individual terms constituting the summation at the left. Physically speaking, when $(EI \cos \phi)$ is negative, it means that the Emf. source is receiving instead of supplying power. In other words, the machine is acting as a motor instead of a generator.

The reader will also note that the subscripts for E and I in Eq. (31) are reversed. This is due to the fact that the supplied power is computed by taking the product of the Emf. and the current flowing through the *internal* (not through the external) part of the circuit. Thus consider **E**₁₂ Fig. 7, Chapter X. This Emf. causes an external current to flow from 1 to 2 or in the direction of the *internal current* **I**₂₁.

The process of computing the left-hand side of Eq. (31) consists of:

- (1) Determining the vector currents in the various branches.
- (2) If a branch uv consists of an Emf. **E**_{uv} then take the current **I**_{vu} = -**I**_{uv} and find the phase angle ϕ between it and the Emf.
- (3) Form the products $E_{uv} I_{vu} \cos \phi$ and add these products algebraically. Their sum must check the right-hand side of Eq. (31).

7. Apparent Power (Power Factor). — Alternating current machinery is rated in volt-amperes (V.A.) or kilovolt-amperes (K.V.A.). This rating is simply obtained by multiplying the terminal voltage of the machine by its current supply. The volt-amperes of a machine is known as the apparent power and is defined as follows:

$$P_a = EI \quad (32)$$

The reason for adopting this mode of rating is that the heating of a given machine, which limits its capacity, is a function of the current supplied and has nothing to do with the power factor of the external load.

The ratio of the average power to the apparent power is known as the Power factor or:

$$\text{P.f.} = P/P_a = EI \cos \phi / EI = \cos \phi \quad (33)$$

8. Reactive Power (Wattless Power).— The term reactive power is introduced in Art. 3 and defined as that double-frequency term, in the instantaneous power expression (Eq. 10b), whose instantaneous value is:

$$P_x = (EI \sin \phi) \sin 2(\omega t + \alpha) \quad (34)$$

Now the accepted definition of reactive power is:

$$P_x = EI \sin \phi \quad (35)$$

This is simply the amplitude of the reactive power wave. Physically it represents the maximum rate at which energy is stored in a circuit.

The definition of reactive power would be satisfactory if it is generally understood that reactive power, so defined, represents the maximum rate at which energy is stored in a circuit.*

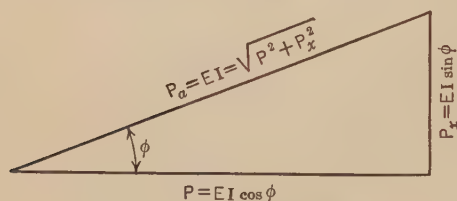


FIG. 6. The power triangle.

9. Relation Between Apparent, Average, and Reactive Power (the Power Triangle).—

The expressions for apparent, average, and reactive power

suggest the existence of a right triangle; which is similar to the voltage and impedance triangles in series circuits, and to the current and admittance triangles in parallel circuits. This triangle (Fig. 6) is called the *power triangle*. It has P_a , P , and P_x for hypotenuse, base, and altitude

* Unfortunately this does not appear to be the case. A questionnaire was prepared by the "National Roumanian Institute for the Study of the Development and Utilization of Sources of Energy," for the "Advisory Committee for the Improvement of Power Factor" of the "International Conference on Large High Voltage Systems." An English edition of this questionnaire is published by the American Institute of Electrical Engineers. Here is the first question asked:

1. Are you of the opinion that the idea of reactive power corresponds to a physical phenomenon or do you consider that this idea is a simple abstract mathematical conception?

If your reply is in the affirmative, what is the physical reality which, you consider, corresponds to the idea of reactive power?

Our answer to these questions is that reactive power, as defined by Eq. (35), represents physically the maximum rate at which energy is stored in a circuit subjected to a sine Emf.

respectively. From the properties of the right triangle the following relations may be derived:

$$\left. \begin{aligned} P_a^2 &= P^2 + P_x^2 & (a) \\ P &= P_a \cos \phi & (b) \\ P_x &= P_a \sin \phi & (c) \end{aligned} \right\} \quad (36)$$

10. Derivation of the Expressions for Apparent, Average, and Reactive Power from the Complex Expressions for E_{12} and I_{12} . — Let the complex expressions for the voltage and current be:

$$\left. \begin{aligned} E_{12} &= E\epsilon^{j\alpha} & (a) \\ I_{12} &= I\epsilon^{j\beta} & (b) \end{aligned} \right\} \quad (37)$$

The product $E_{12}\bar{I}_{12}$ * is:

$$P_c = EI\epsilon^{j(\alpha-\beta)} = EI [\cos (\alpha - \beta) + j \sin (\alpha - \beta)] \quad (38)$$

Now $(\alpha - \beta)$ is the phase angle between E_{12} and I_{12} and is therefore numerically equal to ϕ . Hence Eq. (38) may be written:

$$P_c = EI (\cos \phi + j \sin \phi) \quad (39)$$

The real term in Eq. (39) is the average power, the j term is the reactive power, and the modulus is the apparent power.

We thus have the following rule for computing the three power components in the power triangle:

- (a) Take the conjugate of the current vector.
- (b) Form the complex product $E_{12}\bar{I}_{12}$.
- (c) The modulus of the product is the apparent power, its real part is the average power and its j -part is the reactive power.

11. Power and Energy in Coupled Circuits. — (a) *Instantaneous Power.* — The most general expressions for the current-voltage relation in coupled circuits (Fig. 1d, Chapter XI) with a sine Emf. impressed on the primary are Eqs. (14) Chapter XI, namely:

$$\left. \begin{aligned} E_{m1} \sin (\omega t + \alpha) &= R_1 I_{m1} \sin (\omega t + \beta) \\ &\quad + (X_{L1} - X_{c1}) I_{m1} \sin (\omega t + \beta + 90^\circ) \\ &\quad + X_M I_{m2} \sin (\omega t + \theta + 90^\circ) & (a) \\ 0 &= R_2 I_{m2} \sin (\omega t + \theta) \\ &\quad + (X_{L2} - X_{c2}) I_{m2} \sin (\omega t + \theta + 90^\circ) \\ &\quad + X_M I_{m1} \sin (\omega t + \beta + 90^\circ) & (b) \end{aligned} \right\} \quad (40)$$

* $\bar{I}_{12}$ is the conjugate of I_{12} .

In order to determine the instantaneous power in such a circuit, multiply Eq. (40a) by i_1 and (40b) by i_2 . This gives:

$$\left. \begin{aligned} E_{m1} I_{m1} \sin(\omega t + \alpha) \sin(\omega t + \beta) &= R_1 I_{m1}^2 \sin^2(\omega t + \beta) \\ &+ (X_{L1} - X_{c1}) I_{m1}^2 \sin(\omega t + \beta) \cos(\omega t + \beta) \\ &+ X_M I_{m1} I_{m2} \sin(\omega t + \beta) \cos(\omega t + \theta) \end{aligned} \right\} (a) \quad (41)$$

$$\left. \begin{aligned} 0 &= R_2 I_{m2}^2 \sin^2(\omega t + \theta) + (X_{L2} - X_{c2}) I_{m2}^2 \sin(\omega t + \theta) \\ &\cos(\omega t + \theta) + X_M I_{m1} I_{m2} \cos(\omega t + \beta) \sin(\omega t + \theta) \end{aligned} \right\} (b)$$

Applying the relations $\sin a \sin b = [\cos(a - b) - \cos(a + b)]/2$ and $\sin a \cos b = [\sin(a - b) + \sin(a + b)]/2$ and adding Eq. (41a) and (41b) we have:

$$\begin{aligned} p &= E_1 I_1 [\cos(\beta - \alpha) - \cos(2\omega t + \alpha + \beta)] \\ &= R_1 I_1^2 [1 - \cos 2(\omega t + \beta)] + R_2 I_2^2 [1 - \cos 2(\omega t + \theta)] \\ &\quad + (X_{L1} - X_{c1}) I_1^2 \sin 2(\omega t + \beta) \\ &\quad + (X_{L2} - X_{c2}) I_2^2 \sin 2(\omega t + \theta) + 2 X_M I_1 I_2 \sin(2\omega t + \beta + \theta) \end{aligned} \quad (42)$$

But:

$$\begin{aligned} \cos(2\omega t + \alpha + \beta) &= \cos[2(\omega t + \beta) - (\beta - \alpha)] \\ &= \cos(\beta - \alpha) \cos 2(\omega t + \beta) + \sin(\beta - \alpha) \sin 2(\omega t + \beta) \end{aligned} \quad (43)$$

Substituting Eq. (43) in (42) we have:

$$\left. \begin{aligned} p &= E_1 I_1 \cos(\beta - \alpha) [1 - \cos 2(\omega t + \beta)] \\ &\quad + E_1 I_1 \sin(\beta - \alpha) \sin 2(\omega t + \beta) \end{aligned} \right\} (a)$$

or

$$\left. \begin{aligned} p &= R_1 I_1^2 [1 - \cos 2(\omega t + \beta)] + R_2 I_2^2 [1 - \cos 2(\omega t + \theta)] \\ &\quad + (X_{L1} - X_{c1}) I_1^2 \sin 2(\omega t + \beta) \\ &\quad + (X_{L2} - X_{c2}) I_2^2 \sin 2(\omega t + \theta) \\ &\quad + 2 X_M I_1 I_2 \sin 2(\omega t + \beta + \theta) \end{aligned} \right\} (b) \quad (44)$$

These are the most general expressions for the instantaneous power in two coupled circuits the primary of which is excited by a sine Emf. We shall examine each of the terms in Eqs. (44) in order to gain an insight into their physical meaning.

Eq. (44a) corresponds exactly to Eq. (10b) and represents the power supplied by the Emf. source. The first right-hand term of Eq. (44a) represents the pulsating power which is dissipated in the resistances R_1 and R_2 of the circuit, and is therefore numerically equal to the sum of the first and second terms in Eq. (44b). Thus:

$$\begin{aligned} E_1 I_1 \cos(\beta - \alpha) [1 - \cos 2(\omega t + \beta)] &= R_1 I_1^2 [1 - \cos 2(\omega t + \beta)] \\ &\quad + R_2 I_2^2 [1 - \cos 2(\omega t + \theta)] \end{aligned} \quad (45)$$

The second term of Eq. (44a) represents the rate at which energy is stored in the circuit, and is therefore numerically equal to the sum of the third, fourth and fifth terms in Eq. (44b) or:

$$\begin{aligned} E_1 I_1 \sin(\beta - \alpha) \sin 2(\omega t + \beta) &= (X_{L1} - X_{C1}) I_1^2 \sin 2(\omega t + \beta) \\ &\quad + (X_{L2} - X_{C2}) I_2^2 \sin 2(\omega t + \theta) \\ &\quad + 2 X_M I_1 I_2 \sin 2(\omega t + \beta + \theta) \end{aligned} \quad (46)$$

In Eq. (46) the first term on the right represents the rate at which energy is stored in $\mathfrak{L}_1$ and C_1 , the second term stands for the rate at which energy is stored in $\mathfrak{L}_2$ and C_2 and the third term expresses the rate of energy storage in $\mathfrak{M}$. Any of these terms may be positive, negative, or zero. If the value of a term is positive it means that energy is being stored, if negative it indicates that energy is being returned.

(b) *Average Power.*—The average power supplied to the coupled circuits in the interval of time $t = t_2 - t_1$ is expressed by Eq. (5), where p has the value given in Eqs. (44). Should the average power supplied over a period T be required, then the limits become 0 and T . Multiplying Eqs. (44) by dt and integrating over these limits we have:

$$P = E_1 I_1 \cos(\beta - \alpha) = R_1 I_1^2 + R_2 I_2^2 \quad (47)$$

(c) *Energy.*—The energy supplied to coupled circuits in the time $t = t_2 - t_1$ may be computed by substituting in Eq. (3) the value of $p = ei$ as expressed by Eqs. (44) and integrating; thus:

$$W = \int_{t_1}^{t_2} p \, dt \quad (48)$$

PROBLEMS

1a. A 110-volt D.C. generator supplies a current of 10 amperes to a lamp bank. What is the energy consumed in the course of 5 hours?

1b. The equation for current during the period of establishing a circuit with R and $\mathfrak{L}$ is:

$$i = (E/R) (1 - e^{-Rt/\mathfrak{L}})$$

where E is the continuous impressed Emf., R and $\mathfrak{L}$ are the resistance and inductance of the circuit, and t is the time counted from the instant of establishing the circuit.

(1) Give the expression for the energy supplied to the circuit in an interval $t = t_2 - t_1$.

(2) Assuming that $E = 110$ volts, $R = 10$ ohms, and $\mathfrak{L} = 5$ mh. compute the energy supplied in the course of 0.1 sec. from the time of establishing the circuit.

2a. Give the expression for the instantaneous power supplied by the Emf. source in problem 1b.

2b. The sine voltage and current in an alternating electric circuit have the following instantaneous values:

$$e = 150 \sin (\omega t + 40^\circ)$$

$$i = 20 \sin (\omega t + 10^\circ)$$

(1) Give the expression for the instantaneous power supplied by the source.

(2) What is the average power supplied in a quarter of a cycle from the instant $t = 0$?

(3) What is the average power supplied in one cycle?

3a. Show that for a phase angle ϕ between the sine voltage and current the maximum and minimum ordinates of the instantaneous power curve are respectively:

$$p_{\max.} = EI (\cos \phi + 1) \quad \text{and} \quad p_{\min.} = EI (\cos \phi - 1)$$

3b. The instantaneous values of a power curve vary between +22 K.V.A. and -5 K.V.A. The effective value of the voltage is 1300 volts.

(a) Find the phase angle between the voltage and current.

(b) Compute the power factor.

(c) Write the instantaneous value expressions for the voltage and current waves, assuming that $t = 0$ when $e = 500$ volts and that the slope is positive.

3c. Show that the maximum rate at which energy is stored in a circuit does not necessarily occur at the time when the value of the instantaneous power is a maximum but rather occurs when:

$$p' = EI (\cos \phi + \sin \phi)$$

3d. Show that the maximum rate at which energy is returned from the circuit to the Emf. source occurs at the time when the value of the instantaneous power is:

$$p'' = EI (\cos \phi - \sin \phi)$$

3e. A sine voltage $e = E_m \sin (120 \pi t + 25^\circ)$ causes a current $i = I_m \sin (120 \pi t - 10^\circ)$ determine the instant t when the rate of energy storage is a maximum.

3f. A circuit consisting of 10 ohms resistance is subjected to a 110-volt source. What is the average power dissipated in the circuit?

3g. The power dissipated in a resistance circuit at 220 volts is 550 watts.

(1) Determine the resistance of the circuit.

(2) Find the current flowing through the circuit.

(3) What is the maximum rate at which energy is supplied by the voltage source?

3h. A series circuit consists of a resistance $R = 8$ ohms and an inductance $\mathcal{L} = 0.07$ mh. What is the power dissipated in and the power factor of the circuit if the impressed voltage has an amplitude of 15 volts and a frequency of 5000 cycles per second?

3i. An impedance coil takes 8 amp. and absorbs 320 watts when connected across 110-volt, 60-cycle mains. What non-inductive resistance must be connected in series with this coil in order that it may take the same current from a 220-volt, 25-cycle source?

3j. A voltage $150 \sin (\omega t + 25^\circ)$ when applied to a capacitive load produces a current $7 \sin (\omega t + 60^\circ)$. Determine the average power supplied and the power factor of the load.

3k. A conductance $G = 0.1$ mho and a susceptance $B_L = 0.7$ mho are subjected to a 110-volt source. Compute the power dissipated in the circuit. What is the power factor of this load?

3l. The power dissipated in a parallel circuit consisting of a conductance G and inductive susceptance B_L is 1 kilowatt (Kw.). Moreover, the impressed voltage is 110 volts and the power factor is 0.7. Compute the conductance and susceptance of the circuit.

3m. A voltage $e = 150 \sin 120 \pi t$ supplies a current $i = 10 \sin (\omega t + 25^\circ)$.

- (1) What is the maximum rate at which energy is supplied?
- (2) What is the maximum rate at which energy is returned to the source?
- (3) Determine the average dissipated power.
- (4) Evaluate the power factor of the circuit.

3n. In order to determine whether a machine is running as a motor or as a generator, a two-element oscillograph is properly connected to it. The record shows that the sine waves of current and voltage are:

$$\begin{aligned} e_{12} &= 1500 \sin \omega t \\ i_{12} &= 150 \sin (\omega t - 175^\circ) \end{aligned}$$

Compute the average power, the power factor, and state whether the machine is running as a motor or as a generator.

4a. It is shown in Art. 4 that in the series circuit, Fig. 4a, there exists an interchange of energy between $\mathcal{L}$ and C . Discuss the energy relations when $X_L = X_C$.

4b. A series circuit consists of R , $\mathcal{L}$, C , and a sine Emf. $e = E_m \sin (\omega t + \alpha)$.

- (1) Equate the sum of the instantaneous potential drops, across R , $\mathcal{L}$, and C , to the instantaneous value of the applied Emf.
- (2) Multiply both sides of the equation by the instantaneous value of the current and interpret your result physically.
- (3) Draw the waves for p , p_r , p_L , and p_C , and show that the reactive power in $\mathcal{L}$ is positive when that in C is negative, and vice versa. What is the physical interpretation of this from the standpoint of energy?
- (4) Assume that the circuit is resonant, that is, $X_L = X_C$. Redraw the curves for p_L and p_C and explain the phenomenon of voltage resonance from an energy viewpoint.

4c. A parallel circuit consists of a conductance G , an inductance $\mathcal{L}$, and a sine Emf. $e = E_m \sin (\omega t + \alpha)$.

- (1) Equate the instantaneous value of the total current i supplied by the Emf. source to the sum of the instantaneous values of the branch currents i_g and i_L .
- (2) Multiply both sides of the equation by the instantaneous value of the Emf. and interpret your result physically.
- (3) Transform the product ei to the form given in Eq. (10b) and show that the first term in this equation is identical with the product ei_g and the second term is identical with the product ei_L .
- (4) Draw the instantaneous power curve for one complete cycle and show that the ordinates of this curve are equal to the sum of the corresponding ordinates of the power curves p_g and p_L .

- (5) Compute the average power dissipated in the circuit by integrating the product $p dt$ over a complete cycle and show that this result checks with Eq. (13).
 - (6) Evaluate the energy supplied to the circuit in a quarter of a cycle measured from the instant when $i_L = 0$ and the slope of the curve is negative.
- 4d. A parallel circuit consists of a conductance G , an inductance $\mathcal{L}$, a capacitance C , and a sine Emf. $e = E_m \sin(\omega t + \alpha)$.
- (a) Develop expressions for the instantaneous and average power in this circuit as outlined in problem 4c.
 - (b) Draw the waves for p , p_g , p_L , and p_c , and show that the reactive power in $\mathcal{L}$ is positive when that in C is negative and vice versa. What is the physical interpretation of this from an energy viewpoint?
 - (c) Assume that the circuit is resonant, that is, let $B_L = B_C$. Redraw the curve for p_L and p_c and explain the phenomenon of current resonance from an energy viewpoint.
- 4e. In a resistance circuit the sine voltage and current amplitudes are $E_m = 150$ volts and $I_m = 25$ amp. Moreover, the two waves are in phase. Draw to scale the curve of instantaneous power for one cycle of the impressed voltage.
- 4f. Using a planimeter, find the areas of the power curve plotted in problem 4e, and determine the average power by dividing this area by the base.
- 4g. What is the maximum instantaneous rate at which energy is delivered to a 50-watt, 110-volt lamp on a 60-cycle circuit?
- 4h. Draw to scale the instantaneous power curve in a circuit consisting of an inductance $\mathcal{L} = 0.05$ mh. subjected to a 5000-cycle, 5-volt source.
- 4i. A circuit having a capacitance $C = 5 \times 10^{-4} \mu\text{f.}$ is subjected to a 3-volt, 6×10^5 -cycle source. Draw the curve of instantaneous power.
- 4j. A series circuit consists of a resistanceless inductance $\mathcal{L} = 0.07$ mh. and a capacitor $C = 0.1 \times 10^{-6}$ farad. Draw the curves of instantaneous power for both the inductance and capacitance when this circuit is subjected to an Emf. source having a frequency of 80,000 cycles per second and an amplitude of 10 volts.
- 4k. Add the power curves obtained in problem 4j and show that the net instantaneous power supplied by the source is the same as would be supplied to a circuit consisting of $X'_L = X_L - X_C$. What is the power factor of such a circuit?
- 4l. What would be the maximum amount of energy stored in a quarter of a cycle when an inductance $\mathcal{L} = 0.1$ mh. is subjected to a 25-cycle, 220-volt source?
- 5a. Compute the total power absorbed in the series parallel combination given in problem 3, Chapter X, ($E = 110$ volts).
- 5b. Referring to problem 8c, Chapter X, compute:
- (a) The power dissipated in each branch.
 - (b) The total power dissipated in the circuit:
 - (1) By taking the sum of the powers in the branches.
 - (2) By forming the product $EI \cos \phi$.
6. Referring to problem 7b, Chapter X, determine the average power dissipated in each branch of Fig. 10, Chapter X and show that this is numerically equal to the power supplied by the Emf. sources.

7. Compute the apparent power and power factor of the circuits given in problems 3h, 3i, 3j, 3k, 3m, 3n, 5a, and 5b.

8. Compute the reactive power in the circuits of problems 3h, 3i, 3j, 3k, 3m, 3n, 5a, and 5b.

9. Draw the power triangle for all the problems quoted in problems 7 and 8.

10a. A source of voltage $100 + j50$ volts delivers 7 Kw. to an 80 per cent power factor lagging load. What is the complex expression for the current?

10b. A source of Emf. supplies a current I_2 through two branches of a parallel circuit. If the applied Emf. is $E_{12} = 110 \angle 25^\circ$ and the currents in the two branches are respectively $5 \angle 40^\circ$ and $10 \angle 0^\circ$, determine:

- The apparent, average and reactive power in each branch.
- The apparent, average and reactive power supplied by the source.
- The power factor of the combined load.

10c. Compute the power (average, apparent, and reactive) in the circuits of problems 3h, 3i, 3j, 3k, 3m, 3n, 5a, and 5b by taking the complex product $E\bar{I}$, and check your results with those obtained in problems 7, 8, and 9.

11. Referring to problems 9a to 9c, Chapter XI:

- Compute the instantaneous power supplied by the primary.
- Divide the instantaneous power expression into two parts: one representing power dissipated in the resistances and the other representing the rate at which energy is stored in the circuits.
- Compute the instantaneous power dissipated in the resistances and check its value (for the instant $t = 0$) with the value of the first part of the total instantaneous power.
- Compute (for the same instant) the rate at which energy is stored in each part of the circuit and check this result with the rate at which energy is supplied by the source.
- Plot the waves for instantaneous power in R_1 , R_2 , $\mathcal{L}_1$, $\mathcal{L}_2$, C_1 , C_2 , and $\mathcal{M}$; and show that the sum of their ordinates corresponds, at any instant, with the value $p_1 = e_1 i_1$.
- What do you conclude, from the phase displacement of these various curves, relative to the timing of energy storage in inductively coupled circuits?
- Compute the average power supplied to the two circuits, and check this value with the expression $P = R_1 I_1^2 + R_2 I_2^2$.
- Determine the total energy supplied to the circuit in the course of one-fourth cycle from the instant when i_1 is zero and the slope of its wave is positive.
- What part of the total energy computed above is actually dissipated and what part is stored in the coupled circuits?
- The stored energy in part (9) exists in $\mathcal{L}_1$, C_1 , $\mathcal{L}_2$, C_2 , and $\mathcal{M}$. Give the amount stored in each of these characteristics.

CHAPTER XIII

FOURIER'S SERIES AND HARMONIC ANALYSIS

1. Sines of Multiples of x . — Consider the sine function

$$f(x) = A \sin n(x + \alpha) \quad (1)$$

Where $f(x)$ is the instantaneous value of the sine function, A is its amplitude, α its initial angle, x its variable angle, and n any real number (not zero). Eq. (1) differs from Eq. (1), Chapter III in only one respect — the presence of n as a factor by which the angle $(x + \alpha)$ is multiplied. Eq. (1) therefore represents a sine function of a multiple of x , namely nx .

The graphical interpretation of all the notations defined above (except n) is given, in Art. 3, Chapter III. In order to interpret n graphically consider three sine functions (Fig. 1) in which $n = 1$, $n = 2$, and $n = 3$ respectively. That is, let:

$$\left. \begin{aligned} f_1(x) &= A_1 \sin 1(x + \alpha) & (a) \\ f_2(x) &= A_2 \sin 2(x + \alpha) & (b) \\ f_3(x) &= A_3 \sin 3(x + \alpha) & (c) \end{aligned} \right\} \quad (2)$$

For purposes of terminology we shall use the factor, n , to indicate the *order of harmonic*. Thus if $n = 1$ (see Eq. 2a) the sine function shall be called the *first harmonic* (or fundamental); if $n = 2$ (see Eq. 2b) the sine function is the *second harmonic*; and in general if $n = p$ the sine function shall be known as the *pth harmonic*.

The sine functions given by Eqs. (2) are plotted in Fig. 1. It will be observed that to every cycle of the fundamental there exist two cycles of the second harmonic and three cycles of the third harmonic. In general, to every cycle of the fundamental there exist n cycles of the n th harmonic. But each cycle consists of 2π radians. Therefore to every 2π radians of the fundamental there are $n(2\pi)$ radians of the n th harmonic. Graphically, therefore, n changes the abscissa scale so that the same distance which represents 1 degree or 1 radian on the scale of the fundamental represents respectively n degrees or n radians on the scale of the n th harmonic (see Fig. 1). This relationship of scales is accounted for by multiplying the angle $(x + \alpha)$, which is always measured to the scale of the fundamental, by the number of the harmonic.

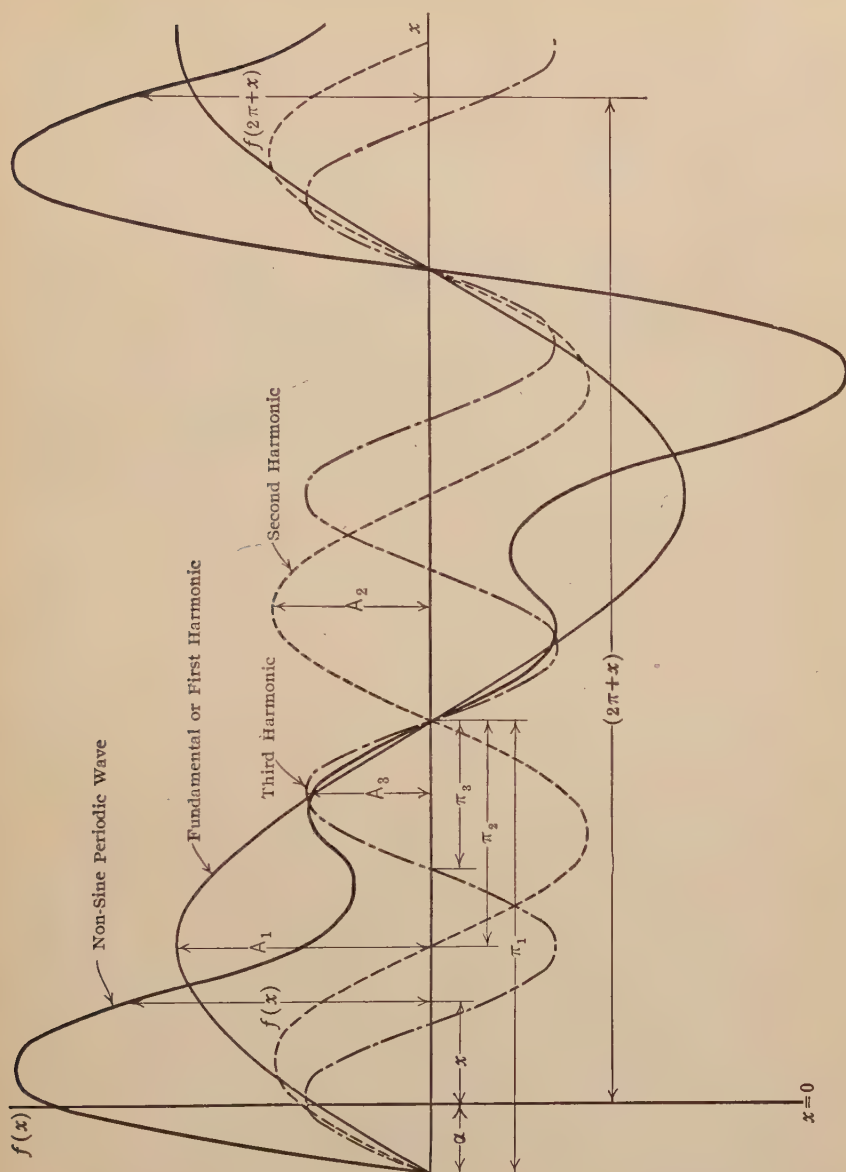


Fig. 1. A non-sine periodic wave and its harmonics.

2. Non-Sine Periodic Waves. — The result of adding corresponding ordinates of the sine waves (Fig. 1) is the irregular wave marked non-sine wave. This curve, although not a sine wave, is a periodic wave because it repeats itself every 2π radians (measured to the scale of the fundamental). In other words, any two ordinates $f(x)$ and $f(2\pi+x)$

that are 2π radians apart are identical. Such a curve is known as a *non-sine periodic wave* or simply a *periodic wave*; and the function which this wave represents is known as a *periodic function*. In order to define a periodic wave mathematically let $f(x)$ be any ordinate corresponding to the abscissa x , and let the period of the wave be 2π radians. Then since any two ordinates that are a period apart are identical we have:

$$f(x) \equiv f(2\pi + x) \quad (3)$$

Eq. (3) is an identity* and holds true for all values of x .

3. Fourier's Theorem. (a) *Statement of the Theorem.* — We have just seen that the sum of three sine waves (the first, second, and third harmonics, Fig. 1) is a periodic non-sine wave. Is the reverse true? Can it be stated that any periodic function $f(x)$ may be represented by a sum of sines of multiples of x ? The electrical profession is greatly indebted to the French physicist Jean Baptiste Joseph Fourier who, in the course of his work on the analytical theory of heat, discovered the fact that:

Any function $f(x)$, which, within an interval, is single-valued, finite, and continuous or (if discontinuous) has only a finite number of discontinuities, may be represented by a series of the form:

$$\left. \begin{aligned} f(x) &= 0.5 A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_n \cos nx + \cdots \\ &\quad + B_1 \sin x + B_2 \sin 2x + \cdots + B_n \sin nx + \cdots \quad (a) \\ \text{or} \quad f(x) &= 0.5 A_0 + \sum_{n=1}^{n=\infty} A_n \cos nx + \sum_{n=1}^{n=\infty} B_n \sin nx \quad (b) \end{aligned} \right\} \quad (4)^\dagger$$

* The student may not fully appreciate the difference between an equality denoted by the symbol $(=)$ and an identity expressed by $(\equiv)$. The equation $5x + 6y = 3x + 4y$ is merely an equality because there are certain values of x and y (for example, $x = 1$ and $y = -1$) which satisfy the equation. But the equation $5x + 6y \equiv 3x + 4y$ is an identity because not only certain values of x and y satisfy it but also any and all values of x and y that one may choose to substitute in the equation will satisfy it.

† The average student is generally afraid to use the summation sign $\sum$. This fear probably arises from the fact that he does not understand its meaning. Let

us state that the expression $\sum_{n=1}^{n=\infty} C_n$ implies the following operations:

- (a) Use 1 as subscript and obtain the term C_1 .
- (b) Place a plus sign after C_1 and use 2 as subscript and obtain $C_1 + C_2$.
- (c) Place a plus sign after C_2 and use 3 as subscript thereby obtaining $C_1 + C_2 + C_3$.
- (d) Continue this process indefinitely and you have

$$\sum_{n=1}^{n=\infty} C_n = C_1 + C_2 + C_3 + \cdots + C_n + \cdots$$

The student can easily see that, if the summations in Eq. (4b) are expanded in this manner, the two expressions (4a) and (4b) would be identical.

where

$$\left. \begin{aligned} A_n &= (1/\pi) \int_0^{2\pi} f(x) \cos nx \, dx & (a) \\ B_n &= (1/\pi) \int_0^{2\pi} f(x) \sin nx \, dx & (b) \end{aligned} \right\} \quad (5)$$

Multiplying Eq. (5b) by j and adding we have:

$$A_n + jB_n = (1/\pi) \int_0^{2\pi} f(x) [\cos nx + j \sin nx] dx = (1/\pi) \int_0^{2\pi} f(x) e^{jnx} dx \quad (5')$$

(b) *Graphic Meaning of the Terms in Fourier's Series.* — The sine and cosine terms entering into Eq. (4) shall be hereafter referred to as the sine and cosine harmonics. The constant term $0.5 A_0$ represents a straight line (Fig. 4d) parallel to the abscissa and at a distance from it equal to the height of a rectangle (of base 2π) whose area is equal to the area under the curve. In order to prove this statement let the series in Eq. (4) be multiplied throughout by dx and integrated from zero to 2π . All the sine and cosine terms vanish and we have:

$$\left. \begin{aligned} \int_0^{2\pi} f(x) dx &= (0.5 A_0) 2\pi & (a) \\ 0.5 A_0 &= (1/2\pi) \int_0^{2\pi} f(x) dx & (b) \end{aligned} \right\} \quad (6)$$

or

But the integral in Eq. (6b) corresponds to the area under the curve and when this is divided by 2π it gives the height of a rectangle whose base is 2π and whose area is equal to the area under the curve.

(c) *Proof of Fourier's Theorem.* — It is outside the scope of this text to prove Fourier's theorem. The reader is referred, for this proof, to treatises and textbooks devoted exclusively to this subject.* It can be easily shown, however, that if the theorem is valid then the coefficients A_n and B_n must have the values expressed by Eqs. (5) or (5'). This proof is given in Appendix I, theorem III.

(d) *Limitations of Fourier's Theorem.* — The limitations, imposed upon the function $f(x)$ that can be expressed by a Fourier's series, must be clearly understood:

- (1) The function must be single-valued; in other words, a function (Fig. 2) which has more than one ordinate corresponding to the same abscissa x can not be represented by a Fourier's series, in the interval $(x_2 - x_1)$ where the function is multi-valued.
- (2) The function must be finite; that is, none of the ordinates $f(x)$ within the interval $(x_2 - x_1)$ can be infinite.

* See for example Byerly's *Fourier's Series and Spherical Harmonics*; Whittaker and Watson's *Modern Analysis*, 3rd edition.

- (3) The function must be continuous or possess a finite number of discontinuities. Graphically this means that the function must not have too many breaks, as is shown in Fig. 3. Just how many discontinuities are allowed within an interval, and the extent of these discontinuities, is something that interests the mathematician.
- (4) It must be observed that nothing is said about periodicity. Thus the function need not be periodic.

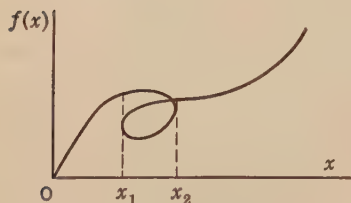


FIG. 2. A multi-valued function.

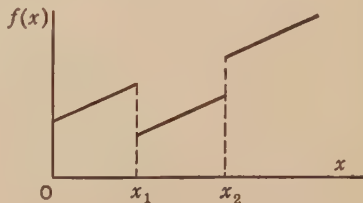


FIG. 3. A discontinuous function.

4. Symmetry.—A periodic function $f(x)$ (of which the period is 2π) is said to possess symmetry when it satisfies one or more of the following conditions:

$$\left. \begin{array}{ll} f(x) \equiv -f(-x) & (a) \\ f(x) \equiv -f(\pi + x) & (b) \\ f(x) \equiv -f(\pi - x) & (c) \end{array} \right\} \quad \left. \begin{array}{ll} f(x) \equiv f(-x) & (d) \\ f(x) \equiv f(\pi + x) & (e) \\ f(x) \equiv f(\pi - x) & (f) \end{array} \right\} \quad (7)$$

In order to visualize the meaning of Eqs. (7) we shall give graphs of functions which satisfy one or more of these symmetry conditions.

(a) If the origin ($x = 0$, Fig. 4) is selected as shown, then the periodic function represented by Fig. 4 would satisfy the symmetry conditions (a), (b), and (f) given in Eq. (7).

(a') The function represented by Fig. 4a satisfies the symmetry condition (a), that is, $f(x) \equiv -f(-x)$, provided the origin is selected as shown. The curve representing this function is such that if one half-cycle is rotated through 180° about the point $x = 0$ as a center, then the two half-waves correspond.

(b) Fig. 4b represents a function possessing the symmetry $f(x) \equiv -f(\pi + x)$. The curves representing such symmetrical functions are characterized by the fact that if a half-cycle is moved along the abscissa through 180° and then folded over the two half-cycles would correspond.

(c) The graph, Fig. 4c, represents a function which satisfies the symmetry $f(x) \equiv -f(\pi - x)$. If a point P is selected at a distance $\pi/2$ from the origin $x = 0$ and the quarter curve is rotated through 180°

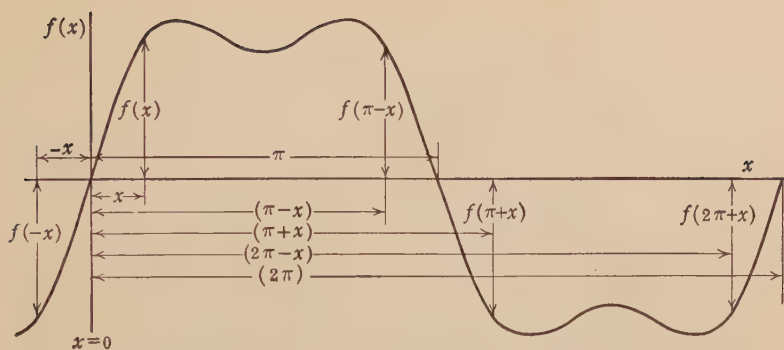


FIG. 4. A periodic function which satisfies the symmetry conditions
 (a) $f(x) \equiv -f(-x)$; (b) $f(x) \equiv -f(\pi+x)$; (c) $f(x) \equiv f(\pi-x)$.

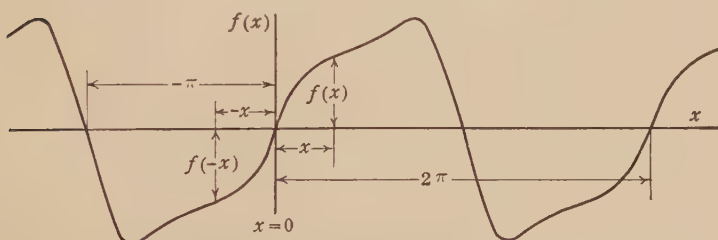


FIG. 4a. A periodic function which satisfies the symmetry $f(x) \equiv -f(-x)$.

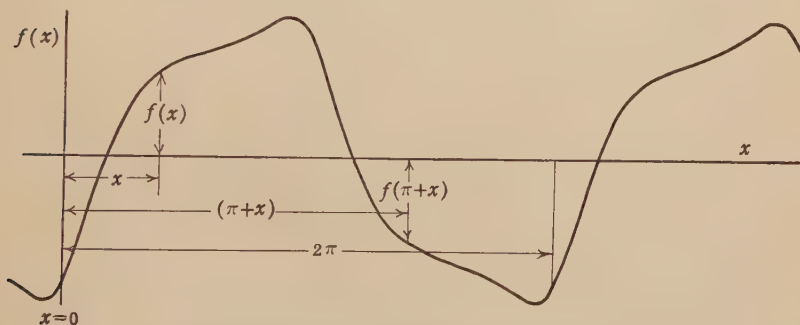


FIG. 4b. A periodic function which satisfies the symmetry $f(x) \equiv -f(\pi+x)$.

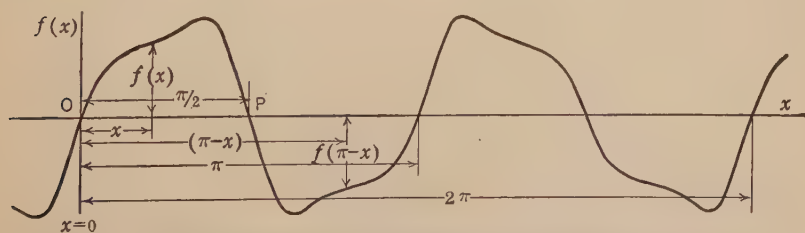


FIG. 4c. A periodic function which satisfies the symmetry $f(x) \equiv -f(\pi-x)$.

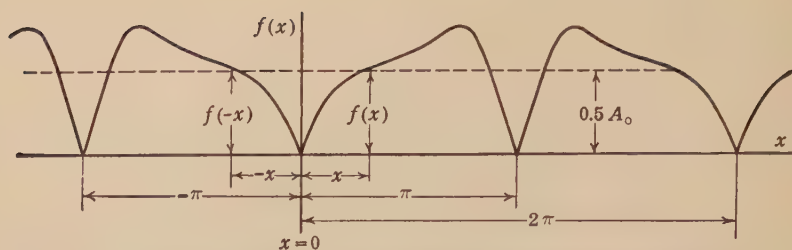


FIG. 4d. A periodic function which satisfies the symmetry $f(x) \equiv f(-x)$.

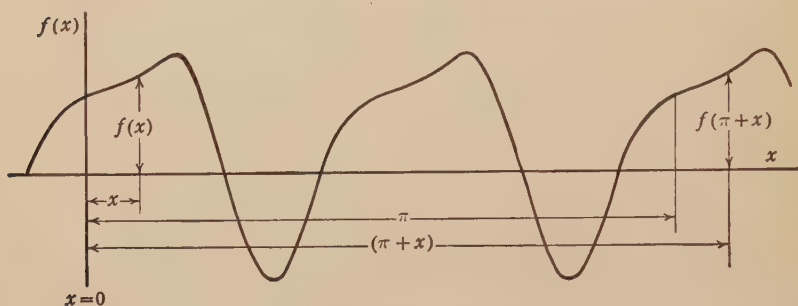


FIG. 4e. A periodic function which satisfies the symmetry $f(x) \equiv f(\pi + x)$.

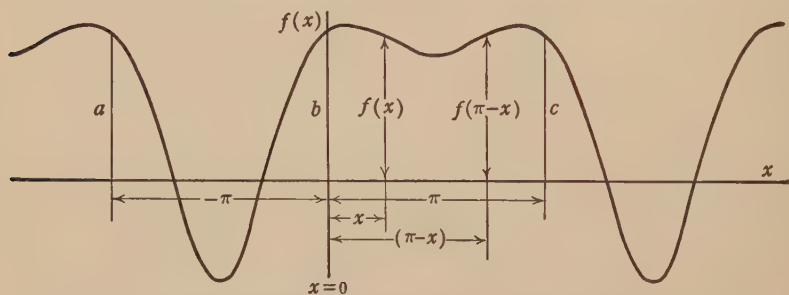


FIG. 4f. A periodic function which satisfies the symmetry $f(x) \equiv f(\pi - x)$.

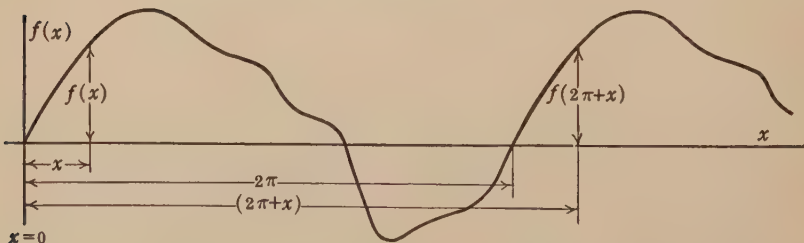


FIG. 4g. A periodic function which does not satisfy any symmetry condition and is therefore an asymmetrical periodic function.

about the point P as a center then the two quarter-cycles would correspond.

(d) The function, drawn in Fig. 4d, satisfies the condition $f(x) \equiv f(-x)$. This function is characterized by the correspondence of the ordinates, to the right and left of $x = 0$.

(e) Fig. 4e is the graph of a function satisfying the symmetry $f(x) \equiv f(\pi + x)$, which is independent of the choice of origin. Strictly speaking, the distance π (Fig. 4e) covers one complete cycle because the function repeats itself every π radians. It is shown in Art. 5c below that this symmetry is satisfied by functions having a constant term and only even harmonics. Consequently a period of the non-sine wave covers a span equal to two periods of its lowest harmonic.

(f) The non-sine wave, Fig. 4f, represents a function which satisfies the symmetry $f(x) \equiv f(\pi - x)$ provided that the origin $x = 0$ is so chosen that if three equal ordinates a , b , and c , are taken, then the distance (along the abscissa) from a to b is equal to the distance from b to c .

(g) Finally, the function represented by Fig. 4e does not satisfy any of the conditions for symmetry and may be called an *asymmetrical periodic function*.

5. Effect of Symmetry on Fourier's Series. — Many of the periodic waves encountered in electrical engineering practice possess some degree of symmetry. Thus Fig. 4a may represent the Emf. wave of an alternator, Fig. 6 is a rectified sine wave, and Fig. 4b stands for the flux distribution in the air-gap of a loaded D.C. machine. Other examples of symmetrical periodic functions occurring in electrical engineering practice will be found among the problems.

In some cases it is convenient, sometimes imperative, to express such functions by a Fourier's series. Fortunately, when symmetry exists, some of the terms in the series vanish. It is essential to know beforehand which terms do vanish due to symmetry, for two reasons:

- (a) It gives one the assurance that no important harmonics have been neglected.
- (b) It saves the time and labor expended in evaluating the amplitudes (A_n and B_n , Eqs. 5) of harmonics that are non-existent.

We shall ascertain which terms are eliminated from the general Fourier's series (Eq. 4) due to each of the symmetry conditions given in Eqs. (7).

(a) *Effect of the Symmetry* $f(x) \equiv -f(-x)$. — Substituting x and $(-x)$ in Eq. (4b) and using the relation $f(x) \equiv -f(-x)$ we have:

$$\left. \begin{aligned} 0.5 A_0 + \sum A_n \cos nx + \sum B_n \sin nx \\ \equiv -[0.5 A_0 + \sum A_n \cos(-nx) + \sum B_n \sin(-nx)] \quad (a) \\ \equiv -[0.5 A_0 + \sum A_n \cos(nx) - \sum B_n \sin nx] \quad (b) \end{aligned} \right\} \quad (8)^*$$

whence: $A_0 + \sum 2 A_n \cos(nx) \equiv 0 \quad (9)$

Since Eq. (9) is an identity it must be satisfied for all value of x . This can be true if and only if:

$$A_0 = A_1 = A_2 = \dots = A_n = 0 \quad (10)$$

This means that the constant term and all the cosine terms in Fourier's series are eliminated whenever the function possesses the symmetry $f(x) \equiv -f(-x)$. Such functions may therefore be represented by the following Fourier's series:

$$f_a(x) = \sum B_n \sin nx \quad (11)$$

(b) *Effect of the Symmetry* $f(x) \equiv -f(\pi + x)$. — Substituting $f(x)$ and $f(\pi + x)$ in Eq. (4b) and using the relation $f(x) \equiv -f(\pi + x)$ we obtain:

$$\begin{aligned} 0.5 A_0 + \sum A_n \cos nx + \sum B_n \sin nx \\ \equiv -[0.5 A_0 + \sum A_n \cos n(\pi + x) + \sum B_n \sin n(\pi + x)] \quad (12) \end{aligned}$$

In Eq. (12) when n is odd, $\cos n(\pi + x) = -\cos nx$ and $\sin n(\pi + x) = \sin nx$. But, when n is even, $\cos n(\pi + x) = \cos nx$ and $\sin n(\pi + x) = \sin nx$. Substituting these values in Eq. (12) we have:

$$\begin{aligned} A_0 + 2 A_2 \cos 2x + 2 A_4 \cos 4x + \dots + 2 A_{2n} \cos 2nx \\ + 2 B_2 \sin 2x + 2 B_4 \sin 4x + \dots + 2 B_{2n} \sin 2nx \equiv 0 \quad (13) \end{aligned}$$

Since Eq. (13) is an identity it must hold true for all values of x . This can be the case if all the constant coefficients are zero, that is:

$$\left. \begin{aligned} A_0 = A_2 = A_4 = A_6 = \dots = A_{2n} = 0 \quad (a) \\ B_2 = B_4 = B_6 = \dots = B_{2n} = 0 \quad (b) \end{aligned} \right\} \quad (14)$$

This means that the constant term A_0 and all even harmonics (sine and cosine) vanish. Hence functions which possess the symmetry $f(x) \equiv -f(\pi + x)$ may be represented by a Fourier's series of the form:

$$f_b(x) = \sum A_p \cos px + \sum B_p \sin px \quad (15)$$

where p is an odd integer.

* The limits of the summation in Eqs. (8) and all succeeding expression are understood to be 1 and ∞ unless otherwise specified.

(c) *Effect of the Symmetry* $f(x) \equiv -f(\pi - x)$. — Substituting (x) and $(\pi - x)$ in Eq. (4b) and equating the resulting two series we have:

$$0.5 A_0 + \sum A_n \cos nx + \sum B_n \sin nx \\ \equiv -[0.5 A_0 + \sum A_n \cos n(\pi - x) + \sum B_n \sin n(\pi - x)] \quad (16)$$

In Eq. (16) when n is an odd integer $\cos n(\pi - x) = -\cos nx$ and $\sin n(\pi - x) = \sin nx$. However when n is an even integer $\cos n(\pi - x) = \cos nx$ and $\sin n(\pi - x) = -\sin nx$.

Substituting these values in Eq. (16) and simplifying we have:

$$A_0 + 2 A_2 \cos 2x + 2 A_4 \cos 4x + \dots + 2 A_{2n} \cos 2nx \\ + 2 B_1 \sin x + 2 B_3 \sin 3x + \dots + 2 B_{(2n+1)} \sin (2n+1)x \equiv 0 \quad (17)$$

Consequently the constant term, the even cosine harmonics, and the odd sine harmonics vanish from Fourier's series due to this type of symmetry. Hence the series for functions which satisfy the symmetry $f(x) \equiv -f(\pi - x)$ is:

$$f_c(x) = \sum A_p \cos px + \sum B_q \sin qx \quad (18)$$

where p is an odd integer and q is an even integer.

(d) *Effect of the Symmetry* $f(x) \equiv f(-x)$. — Following the procedure outlined in paragraph (a) we find that the existence of this symmetry results in the elimination of all the sine terms in Fourier's series. The series thus becomes:

$$f_d(x) = 0.5 A_0 + \sum A_n \cos nx \quad (19)$$

(e) *Effect of the Symmetry* $f(x) \equiv f(\pi + x)$. — Proceeding as in paragraph (b) we find that the effect of this type of symmetry is the vanishing of all odd sine and cosine harmonics. The series thus becomes:

$$f_e(x) = 0.5 A_0 + \sum A_q \cos qx + \sum B_q \sin qx \quad (20)$$

where q is an even integer.

(f) *Effect of the Symmetry* $f(x) \equiv f(\pi - x)$. — It can be shown, in a manner similar to that given in paragraph (c), that the effect of this symmetry is to eliminate all the odd cosine harmonics and all the even sine harmonics, thus reducing Fourier's series to the form:

$$f_f(x) = 0.5 A_0 + \sum A_p \cos px + \sum B_p \sin px \quad (21)$$

where p is an odd integer and q is an even integer.

(g) *Effect of the Existence of More Than One Symmetry Condition.* — If a periodic function satisfies more than one symmetry condition then the series representing this function does not comprise any of the terms

that are eliminated through the existence of these conditions. This statement can best be illustrated by an example.

Consider the function represented by Fig. (4). This function satisfies the following symmetry conditions:

$$\left. \begin{aligned} f(x) &\equiv -f(-x) & (a) \\ f(x) &\equiv -f(\pi + x) & (b) \\ f(x) &\equiv f(\pi - x) & (f) \end{aligned} \right\} \quad (22)$$

Through symmetry (a) the series reduces to the form $f(x) = \sum B_n \sin nx$. Through symmetry (b) all the even harmonics vanish and the series becomes $f(x) = \sum B_p \sin px$ (p is an odd integer). Symmetry (f) is not independent because it can be derived from conditions (a) and (b). Thus the series representing Fig. 4a is:

$$f_g(x) = \sum B_p \sin px \quad (p \text{ any odd integer}) \quad (23)$$

The deductions, arrived at in this article, are recapitulated in Table I for purposes of reference.

The series representing periodic functions which satisfy the first three types of symmetry do not have a constant term (see Table I). This means that the areas under the positive portion of the curve are equal to those under the negative portion (examine Figs. 4, 4a, 4b and 4c). This is not true of functions which satisfy symmetries d , e and f .

6. Effect of Symmetry on the Integral Expression Eq. (5'). — The problem of evaluating the constants A_n and B_n in a Fourier's series is, quite often, a tedious and perhaps an exasperating task. Hence any theoretical considerations which result in lightening this work are well worth having. We shall study the effect of each of the symmetries on the integrals in Eq. (5').

(a) *Effect of Symmetry (a).* — The functions represented by Figs. 4 and 4a are periodic in 2π . Hence the integral (Eq. 5') may be written:

$$\begin{aligned} A_n + jB_n &= (1/\pi) \int_{-\pi}^{+\pi} f(x) \epsilon^{jnx} dx \\ &= (1/\pi) \left[\int_{-\pi}^0 f(y) \epsilon^{jny} dy + \int_0^{\pi} f(x) \epsilon^{jnx} dx \right] \end{aligned} \quad (24)$$

Let $y = (-x)$; then $f(y) = f(-x)$; $dy = -dx$. As to the limits: when $y = -\pi$, $x = \pi$; and when $y = 0$, $x = 0$. Substituting these values in Eq. (24) we have:

$$\left. \begin{aligned} A_n + jB_n &= (1/\pi) \left[- \int_{\pi}^0 f(-x) \epsilon^{-jnx} dx + \int_0^{\pi} f(x) \epsilon^{jnx} dx \right] & (a) \\ &= (1/\pi) \left[\int_0^{\pi} f(-x) \epsilon^{-jnx} dx + \int_0^{\pi} f(x) \epsilon^{jnx} dx \right] & (b) \end{aligned} \right\} \quad (25)$$

Substituting the symmetry condition $f(x) \equiv -f(-x)$ in Eq. (25b) we obtain:

$$A_n + jB_n = (1/\pi) \left[\int_0^\pi f(x) \{ \epsilon^{jnx} - \epsilon^{-jnx} \} dx \right] \quad (26)$$

$$\text{But:} \quad \epsilon^{jnx} - \epsilon^{-jnx} = 2j \sin nx \quad (27)$$

Therefore

$$A_n + jB_n = (2j/\pi) \int_0^\pi f(x) \sin nx dx \quad (28)$$

$f(x)$ is always real. Hence the expression at the right in Eq. (28) is always imaginary. Equating reals and imaginaries we have:

$$\left. \begin{aligned} A_n &= 0 & (a) \\ B_n &= (2/\pi) \int_0^\pi f(x) \sin nx dx & (b) \end{aligned} \right\} \quad (29)$$

Table I shows that symmetry (a) results in the elimination of all the cosine harmonics in Fourier's series. Eq. (29) gives an additional confirmation of this fact; and further states that for functions which satisfy symmetry (a) it is not necessary (in order to find the constants B_n) to carry the integration over a complete cycle, as required by Eq. (5b), but simply to integrate over half a cycle and multiply the result by $(2/\pi)$. The significance of this fact becomes more apparent when the work involved in actually evaluating the constants A_n and B_n is outlined in Art. 7 below.

(b) *Effect of Symmetry (b).* — Dividing the integral Eq. (5') into two parts we have:

$$A_n + jB_n = (1/\pi) \left[\int_0^\pi f(x) \epsilon^{jnx} dx + \int_\pi^{2\pi} f(y) \epsilon^{jny} dy \right] \quad (30)$$

Let $y = (\pi + x)$; then $f(y) = f(\pi + x)$, $dy = dx$. Moreover when $y = \pi$, $x = 0$ and when $y = 2\pi$, $x = \pi$. Substituting these equalities in (30) we have:

$$\left. \begin{aligned} A_n + jB_n &= (1/\pi) \left[\int_0^\pi f(x) \epsilon^{jnx} dx + \int_0^\pi f(\pi + x) \epsilon^{jn(\pi+x)} dx \right] & (a) \\ &= (1/\pi) \left[\int_0^\pi f(x) \epsilon^{jnx} dx + \int_0^\pi f(\pi + x) \epsilon^{jn\pi} \epsilon^{jnx} dx \right] & (b) \end{aligned} \right\} \quad (31)$$

Now referring to Table I, it will be observed that Fourier's series for functions which satisfy symmetry (b) contains exclusively odd harmonics. Hence Eq. (31b) becomes:

$$A_p + jB_p = (1/\pi) \left[\int_0^\pi f(x) \epsilon^{jp x} dx + \int_0^\pi f(\pi + x) \epsilon^{jp\pi} \epsilon^{jp x} dx \right] \quad (32)$$

$$\text{But} \quad \epsilon^{jp\pi} = \cos p\pi + j \sin p\pi = -1 \quad (33)$$

Substituting Eq. (33) in Eq. (32) and also substituting the symmetry condition $f(x) \equiv -f(\pi + x)$ we have:

$$A_p + jB_p = (2/\pi) \int_0^\pi f(x) \epsilon^{jpx} dx \quad (34)$$

whence, by equating reals and imaginaries we obtain:

$$\left. \begin{aligned} A_p &= (2/\pi) \int_0^\pi f(x) \cos px \, dx & (a) \\ B_p &= (2/\pi) \int_0^\pi f(x) \sin px \, dx & (b) \end{aligned} \right\} \quad (35)$$

Eqs. (35) state that if a periodic function satisfies symmetry (b) (see Figs. 4 and 4b) then the values of the constants A_n and B_n can be evaluated by carrying the integration over only half a cycle instead of a complete cycle and multiplying the resulting integral by $(2/\pi)$.

(c) *Effect of Symmetry (c).* — We shall leave it to the reader to show that the symmetry $f(x) \equiv -f(\pi - x)$ does not result in any simplification of the integral Eq. (5').

(d) *Effect of Symmetry (d).* — Substituting in Eq. (25b) the symmetry condition $f(x) \equiv f(-x)$ we have:

$$A_n + jB_n = (1/\pi) \left[\int_0^\pi f(x) \{ \epsilon^{jnx} + \epsilon^{-jnx} \} dx \right] \quad (36)$$

$$\text{But} \quad \epsilon^{jnx} + \epsilon^{-jnx} = 2 \cos nx \quad (37)$$

$$\text{Therefore} \quad A_n + jB_n = (2/\pi) \int_0^\pi f(x) \cos nx \, dx \quad (38)$$

Because $f(x)$ is always real the integral is always real. Equating reals and imaginaries we have:

$$\left. \begin{aligned} A_n &= (2/\pi) \left[\int_0^\pi f(x) \cos nx \, dx \right] & (a) \\ B_n &= 0 & (b) \end{aligned} \right\} \quad (39)$$

Table I shows that Fourier's series for periodic functions which satisfy symmetry (d) does not contain any sine harmonics. Eq. (39) furnishes an additional proof of this fact (because $B_n = 0$). Moreover, the amplitudes A_n of the cosine harmonics (contained in the series) may be computed from Eq. (39a) instead of Eq. (5a).

(e) *Effect of Symmetry (e).* — Table I shows that the series representing functions which satisfy this symmetry contain no odd harmonics.

Hence Eq. (31b) may be written:

$$A_q + jB_q = (1/\pi) \left[\int_0^\pi f(x) \epsilon^{jqx} dx + \int_0^\pi f(\pi + x) \epsilon^{jq\pi} \epsilon^{jqx} dx \right] \quad (40)$$

But $\epsilon^{jq\pi} = \cos q\pi + j \sin q\pi = 1$ (41)

Substituting the symmetry condition $f(x) \equiv f(\pi + x)$ and $\epsilon^{jq\pi} = 1$ in Eq. (40) we have:

$$A_q + jB_q = (2/\pi) \int_0^\pi f(x) \epsilon^{jqx} dx \quad (42)$$

whence:

$$\left. \begin{aligned} A_q &= (2/\pi) \int_0^\pi f(x) \cos qx dx & (a) \\ B_q &= (2/\pi) \int_0^\pi f(x) \sin qx dx & (b) \end{aligned} \right\} \quad (43)$$

We thus conclude that Eqs. (5) can be replaced by Eqs. (43) in computing the constants A_n and B_n in the Fourier's series which represents functions possessing symmetry (e).

(f) *Effect of the Existence of More Than One Symmetry.* — The function represented by Fig. 4 satisfies symmetries (a), (b) and (f). Symmetry (a) is dependent (it can be derived from symmetries b and f). We shall therefore consider only symmetries (b) and (f).

We have seen that the symmetry (b) reduces the integral Eq. (5') to Eq. (34). The question is whether the joint existence of symmetries (b) and (f) result in a further simplification of the integral. Let the integral Eq. (34) be written in the form:

$$A_p + jB_p = (2/\pi) \left[\int_0^{\pi/2} f(x) \epsilon^{jpx} dx + \int_{\pi/2}^\pi f(y) \epsilon^{jpy} dy \right] \quad (44)$$

Now let $y = (\pi - x)$. Then $f(y) = f(\pi - x)$ and $dy = -dx$. Again when $y = \pi/2$, $x = \pi/2$; and when $y = \pi$, $x = 0$. Substituting these values in Eq. (44) we have:

$$A_p + jB_p = (2/\pi) \left[\int_0^{\pi/2} f(x) \epsilon^{jpx} dx - \int_{\pi/2}^0 f(\pi - x) \epsilon^{-jpx} \epsilon^{jp\pi} dx \right] \quad (45)$$

But $\epsilon^{jp\pi} = -1$, and by the symmetry (f) we have $f(\pi - x) \equiv f(x)$. Substituting these values in Eq. (45) we have:

$$A_p + jB_p = (2/\pi) \left[\int_0^{\pi/2} f(x) [\epsilon^{jpx} - \epsilon^{-jpx}] dx \right] \quad (46)$$

or $A_p + jB_p = (4j/\pi) \int_0^{\pi/2} f(x) \sin px dx$ (47)

Equating reals and imaginaries we have:

$$\left. \begin{aligned} A_p &= 0 & (a) \\ B_p &= (4/\pi) \int_0^{\pi/2} f(x) \sin px \, dx & (b) \end{aligned} \right\} \quad (48)$$

Table I shows that due to symmetries (b) and (f), Fourier's series reduces to the following form:

$$f(x) = \sum B_p \sin px \quad (49)$$

Eq. (48) confirms this result and further states that the constant B_p can be determined by simply performing the integration over a quarter of a cycle and multiplying the result by $(4/\pi)$.

As these facts are of practical importance to persons dealing with harmonic analysis, they are recapitulated in Table I. Interested readers are urged to make use of this table. Some of the facts revealed therein are not generally recognized.

In the list of problems covering this and the preceding article the student is asked to extend this table to functions which simultaneously satisfy more than one of the symmetry conditions. The reader whose work forces upon him the necessity of dealing with Fourier's series may profit by constructing such a table.

7. Expressing a Function by a Fourier's Series.—Generally speaking, functions which satisfy the Fourier conditions may be divided into two classes:

- (a) Explicitly defined functions.
- (b) Graphically defined functions.

(a) *Explicitly Defined Functions.*—This class includes functions which are or may be explicitly defined in terms of x . Examples of this class of functions are those represented by Figs. (5), (6), (7), (8), (14), (15), (16), and (17). In each case $f(x)$ is explicitly defined in terms of x within the intervals over which the function is known. Thus the function represented by the curve Fig. (5) is defined by Eq. (50). Fig. 6 represents a rectified sine wave of which the equation is $f(x) = \sin(x/2)$. In other words, the functional relation between x and $f(x)$ in this class of function is given or may be found.

(b) *Graphically Defined Functions.*—This class comprises functions which are given either in the form of a set of data in two variables x and $f(x)$, or as an oscillogram or a graph. The majority of functions commonly met with in electrical engineering practice are of this type. Thus the torque of a motor as a function of its speed, the voltage of a generator as a function of its excitation, the flux density in the air-gap

of a machine as a function of the distance along the air-gap are illustrations. These and many other examples constitute the group of functions falling in this class.

In general, the procedure of expressing a function by a Fourier's series consists of:

- (1) Determining whether the function satisfies Fourier's conditions, namely, whether it is single-valued, finite, and continuous.
- (2) Ascertaining whether the function is periodic.
- (3) Examining the periodic function for symmetry.
- (4) Actually evaluating A_n and B_n and substituting their values in the series.

The first three steps are fully discussed and illustrated in the preceding articles. The last step in the procedure is different for the two classes of functions given above. It is therefore necessary to discuss this topic under two cases:

Case (a): $f(x)$ explicitly defined in terms of x .

Case (b): $f(x)$ graphically defined in terms of x .

These two cases are best illustrated by solving specific problems.

Case (a) $f(x)$ Explicitly Defined in Terms of x .

Problem 1. The function whose graph is Fig. 5 may be defined as follows:

$$\left. \begin{aligned} f(x) &= ax \text{ in the interval } 0 \leq x \leq 0.5\pi & (a) \\ f(x) &= a(\pi - x) \text{ in the interval } 0.5\pi \leq x \leq 1.5\pi & (b) \\ f(x) &= a(x - 2\pi) \text{ in the interval } 1.5\pi \leq x \leq 2\pi & (c) \end{aligned} \right\} \quad (50)$$

It is required to express this function by a Fourier's series and to evaluate the terms A_n and B_n in the series.

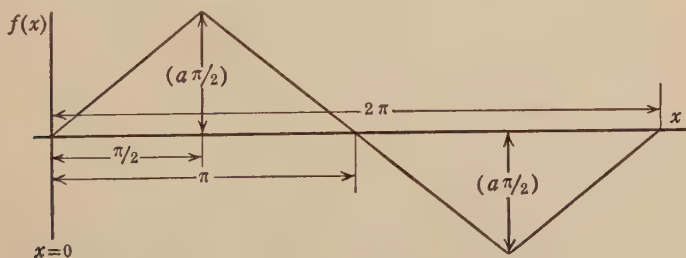


FIG. 5. An explicitly defined periodic function.

Solution. — If the origin ($x = 0$) is selected as shown in Fig. 5, then the function would possess symmetries (b) and (f) (Table I). Hence the Fourier's series representing this function is given by Eq. (23):

$$f(x) = \sum B_p \sin px \quad (p \text{ is an odd integer}) \quad (23)$$

Again, since the function satisfies symmetries (b) and (f), B_p can be evaluated by the use of Eq. (48):

$$B_p = (4/\pi) \int_0^{\pi/2} f(x) \sin px \, dx = (4/\pi) \int_0^{\pi/2} ax \sin px \, dx \quad (51)$$

Integrating Eq. (51), substituting limits, and simplifying we have:

$$B_p = (4 a/\pi p^2) \sin (p\pi/2) \quad (52)$$

Substituting this value of B_p in Eq. (23) and expanding the summation we have:

$$f(x) = (4 a/\pi) \left[(1/1^2) \sin x - (1/3^2) \sin 3x + (1/5^2) \sin 5x - (1/7^2) \sin 7x + \dots \right] \quad (53)$$

Problem 2. The graph (Fig. 6) represents a rectified sine wave which may be defined as follows:

$$f(x) = E_m \sin (x/2) \text{ in the interval } 0 \leq x \leq 2\pi \quad (54)$$

It is required to express the function by a Fourier's series and to evaluate the constants of the series.

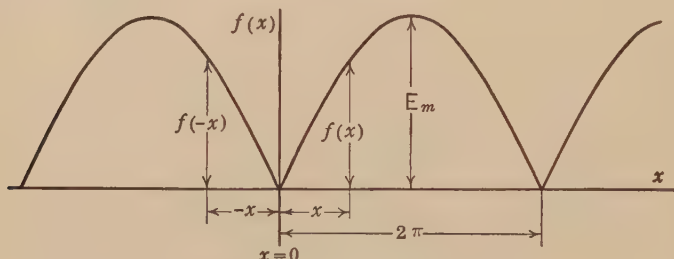


FIG. 6. A rectified sine wave (explicitly defined function).

Solution. — This function possesses the symmetry $f(x) \equiv f(-x)$. Hence it can be represented by a Fourier's series of the form (see Table I and Eq. 19):

$$f(x) = 0.5 A_0 + \sum A_n \cos nx \quad (n \text{ is any integer}) \quad (19)$$

Moreover, since it satisfies symmetry (d), A_n can be evaluated by the use of the following equation (see Table I):

$$A_n = (2/\pi) \int_0^\pi f(x) \cos nx \, dx = (2/\pi) \int_0^\pi E_m (\sin x/2) \cos nx \, dx \quad (55)$$

whence:

$$A_n = 4 E_m / [\pi (1 - 4 n^2)] \quad (56)$$

Therefore the Fourier's series representing the rectified sine wave Fig. 6 is:

$$f(x) = (4 E_m / \pi) \left[0.5 - (1/3) \cos x - (1/15) \cos 2x - (1/35) \cos 3x - \dots \right] \quad (57)$$

Problem 3. Fig. 7 represents a periodic function which may be defined as follows:

$$\left. \begin{aligned} f(x) &= cx \text{ in the interval } 0 \leq x \leq \pi & (a) \\ f(x) &= c\pi \text{ in the interval } \pi \leq x \leq 2\pi & (b) \end{aligned} \right\} \quad (58)$$

It is required to express this function by a Fourier's series and to evaluate the constants A_n and B_n of the series.

Solution. — Since this function possesses none of the symmetries given in Table I, it is represented by the general Fourier's series (Eq. 4b). The coefficients A_n and

B_n may most easily be determined by the use of Eq. (5'). As the function has two different values in the intervals 0 to π and π to 2π it will be necessary to divide the integral Eq. (5') into two parts. We thus have:

$$A_n + jB_n = (1/\pi) \left[\int_0^\pi cx \epsilon^{jnx} dx + \int_\pi^{2\pi} c\pi \epsilon^{jnx} dx \right] \quad (59)$$

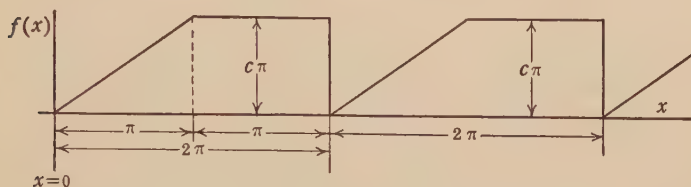


FIG. 7. An explicitly defined periodic function.

Integrating Eq. (59) we have:

$$A_n + jB_n = \left[(c\epsilon^{jnx}/\pi n^2)(1 - jnx) \right]_0^\pi + \left[c\epsilon^{jnx}/jn \right]_\pi^{2\pi} \quad (60)$$

Substituting limits, simplifying, and equating reals and imaginaries we have:

$$\left. \begin{array}{l} \text{For } n = q = \text{an even integer } A_q = 0; B_q = -(c/q) \quad (a) \\ \text{For } n = p = \text{an odd integer } A_p = -(2c/p^2\pi); B_p = -(c/p) \quad (b) \end{array} \right\} \quad (61)$$

In order to determine the term $0.5 A_0$ we must return to Eq. (59) and substitute $n = 0$. This gives:

$$A_0 = (1/\pi) \left[\int_0^\pi cx dx + \int_\pi^{2\pi} c\pi dx \right] = 1.5 c\pi \quad (62)$$

Thus the function represented by the graph (Fig. 7) may be expressed by the following Fourier's series:

$$f(x) = 0.75 c\pi - \sum (2c/p^2\pi) \cos px - \sum (c/n) \sin nx \quad (63)$$

(p an odd integer and n any integer)

Problem 4. — Since the applicability of Fourier's series is not limited to periodic functions we shall select, as another illustrative problem, an aperiodic function. Thus let it be required to find the Fourier's series representing the straight line in Fig. 8. The equation of this line is:

$$f(x) = a + bx \text{ in the interval } 0 \leq x \leq 2\pi \quad (64)$$

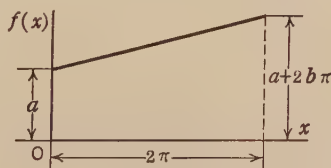


FIG. 8. An aperiodic function.

Solution. — The general expression for Fourier's series is given by Eq. (4) and the constants can, in this case, be best evaluated by using Eq. (5'):

$$A_n + jB_n = (1/\pi) \int_0^{2\pi} (a + bx) \epsilon^{jnx} dx \quad (65)$$

Integrating this expression we have:

$$A_n + jB_n = (1/\pi) \left[(a\epsilon^{jn\pi}/jn) + (b\epsilon^{jn\pi}/\pi n^2)(1 - jn\pi) \right]_0^{2\pi} \quad (66)$$

Substituting limits, simplifying, and equating reals and imaginaries we obtain:

$$\left. \begin{aligned} A_n &= 0 & (a) \\ B_n &= -(2b/\pi n) & (b) \end{aligned} \right\} \quad (67)$$

In order to determine A_0 we have to return to Eq. (65), substitute $n = 0$ and integrate, thereby obtaining:

$$A_0 = 2(a + b\pi) \quad (68)$$

Hence the Fourier's series representing the straight line Fig. 8 is:

$$f(x) = (a + b\pi) - \sum (2b/\pi n) \sin nx \quad (n \text{ any integer}) \quad (69)$$

Naturally under ordinary conditions Eq. (64) is a much simpler way of expressing $f(x)$ than Eq. (69). But for some purposes of circuit analysis a Fourier's series may be more satisfactory.

It must be here emphasized that the function $f(x)$ is defined only in the interval from 0 to 2π . Consequently x in Eq. (69) must not exceed 2π .

This concludes the illustrative problems covering functions that are or may be explicitly defined. Simple functions have been chosen in order to illustrate the method without obscuring it with unnecessarily complicated mathematics. The method is general and applies to any explicitly defined function provided it complies with the requirements mentioned in Art. 3. Other problems, mostly drawn from electrical engineering practice, are given at the end of this chapter.

Case (b). $f(x)$ Graphically Defined in Terms of x .—In this case $f(x)$ is not an explicitly defined function of x . Therefore the values of A_n and B_n can not be evaluated by using the integrals (Eq. 5). In fact there exists no accurate means of determining these constants. We therefore have to resort to approximations. One approximation naturally suggests itself. Why not replace the integral by a summation which in the limit becomes the integral? This is exactly what is done.

If $f(x)$ is uniformly continuous in the interval $0 \leq x \leq 2\pi$ then by the definition of an integral:

$$\lim_{\Delta x \rightarrow 0} (1/\pi) \sum_{u=0}^{u=S} f(u \Delta x) \epsilon^{jn(u \Delta x)} \Delta x = (1/\pi) \int_0^{2\pi} f(x) \epsilon^{jnx} dx \quad (70)$$

where (see Fig. 9)

S is the total number of equal parts into which the interval 0 to 2π is divided;

u is the serial number of the points constituting the limits of these parts;

n is the number of harmonic;

Δx is the length of any one of the equal parts into which the interval is divided.

Hence:

$$\Delta x = 2\pi/S \text{ or } 1/\pi = 2/S \Delta x \quad (71)$$

Substituting this value of $1/\pi$ in the left-hand side of Eq. (70) we have:

$$\lim_{\Delta x \rightarrow 0} (2/S) \sum_{u=0}^{u=S} f(u \Delta x) \epsilon^{jn(u \Delta x)} = (1/\pi) \int_0^{2\pi} f(x) \epsilon^{jnx} dx \quad (72)$$

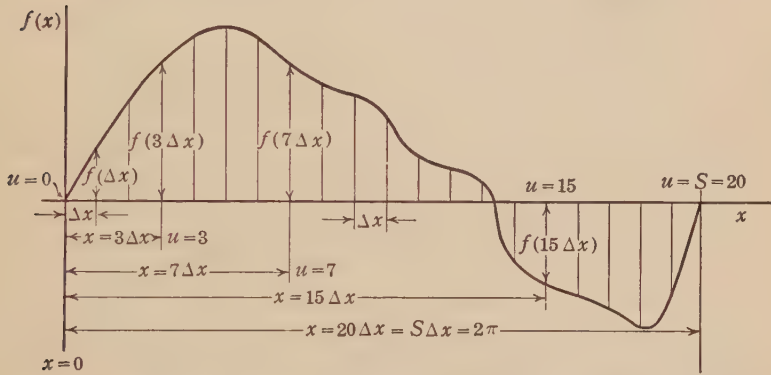


FIG. 9. Evaluation of the constants A_n and B_n for a graphically defined function.

Consequently, for functions which are not explicitly defined in terms of x , we can replace the integral (Eq. 5') by its approximate value, the summation given in Eq. (72). We thus have:

$$A_n + jB_n = (2/S) \sum_{u=0}^{u=S} f(u \Delta x) \epsilon^{jn(u \Delta x)} \quad (73)$$

Expanding Eq. (73) and equating reals and imaginaries we obtain:

$$\left. \begin{aligned} A_n &= (2/S) \sum_{u=0}^{u=S} f(u \Delta x) \cos n(u \Delta x) \quad (a) \\ B_n &= (2/S) \sum_{u=0}^{u=S} f(u \Delta x) \sin n(u \Delta x) \quad (b) \end{aligned} \right\} \quad (74)$$

If the function is periodic and satisfies symmetries (a), (b), (d), or (e), then the summation may be carried over half a cycle only and we have:

$$\left. \begin{aligned} A_n &= (4/S) \sum_{u=0}^{u=S/2} f(u \Delta x) \cos n(u \Delta x) \quad (a) \\ B_n &= (4/S) \sum_{u=0}^{u=S/2} f(u \Delta x) \sin n(u \Delta x) \quad (b) \end{aligned} \right\} \quad (75)$$

If the function satisfies conditions (b) and (f) simultaneously, then Eqs. (74) may be still further simplified to the following form:

$$\left. \begin{aligned} A_n &= 0 & (a) \\ B_p &= (8/S) \sum_{n=0}^{u=S/4} f(u \Delta x) \sin p(u \Delta x) & (b) \end{aligned} \right\} \quad (76)$$

These equations furnish a statement of the successive operations to be done in order to express a function $f(x)$ by a Fourier's series when that function $f(x)$ is not explicitly but is graphically defined through a curve or an oscillogram in which the abscissas correspond to x and the ordinates to $f(x)$. The following is an outline of these operations:

- (1) Examine the curve for periodicity and symmetry and
 - (a) Write the series to include only such terms that are not eliminated through symmetry.
 - (b) Decide whether Eq. (74) or (75) or (76) has to be used in evaluating A_n and B_n and write that equation.
- (2) Divide the abscissa into S equal parts such that the width of each part is $\Delta x = 2\pi/S$. (See Fig. 9)
- (3) Number the points of division 0, 1, 2, 3 . . . S .
- (4) Construct a table (for each harmonic n) with the following column headings (see Tables IX to XV in Appendix II):
 - (a) Number of division u
 - (b) $u \Delta x = u(2\pi/S) = x$
 - (c) $\cos n(u \Delta x) = \cos x$
 - (d) $\sin n(u \Delta x) = \sin x$
 - (e) $f(u \Delta x) = f(x)$
 - (f)* the product $f(u \Delta x) \cos n(u \Delta x) = f(x) \cos x$
 - (g)* the product $f(u \Delta x) \sin n(u \Delta x) = f(x) \sin x$
- (5) Look up the sines and cosines of $n(u \Delta x)$ and enter them in the table, and be careful to use the proper signs + or -.
- (6) Scale off the values $f(u \Delta x)$ and enter them in the table. Call + any ordinate which is above the abscissa and - any ordinate which is below it.
- (7) Form the products required by columns (f) and (g) and enter them in the *plus* column if they are *positive* and in the *minus* column if they are *negative*.
- (8) Add the numbers tabulated in each of the four columns and call the sums k_1, k_2, k_3 and k_4 respectively.

* The last two columns must be divided into two parts one for the positive and the other for the negative products.

- (9) Find $(k_1 - k_2) = k_a$, which might be negative or positive depending upon whether k_1 is greater or less than k_2 .
- (10) Find $(k_3 - k_4) = k_b$ which might be negative or positive.
- (11) Evaluate $A_n = 2 k_a/S$. (See Eqs. 74 to 76.)
- (12) Evaluate $B_n = 2 k_b/S$. (See Eqs. 74 to 76.)
- (13) As to the constant term $(0.5 A_0)$, it will be recalled that this represents the height of a rectangle whose base is equal to the interval over which the function is defined and whose area is equal to the net area under the curve. Therefore in order to find $0.5 A_0$ graphically simply find the net area under the curve by means of a planimeter and divide this area by the length of the interval $S \Delta x$. The student is warned not to assume $S \Delta x = 2 \pi$ because graphically it represents a length which should be measured in inches or centimeters depending upon whether the planimeter registers the area in square inches or square centimeters.
- (14) Having evaluated all the constants of the series, substitute these constants in the expression written in operation 1a and obtain the desired Fourier's series.

The reader need not be reminded that the true values of A_n and B_n can only be found by evaluating the integral Eq. (5') and that the accuracy of their evaluation through the summation given in Eqs. (74) to (76) is limited. The accuracy of the result depends upon the number u of ordinates taken as well as upon the shape of the wave to be analyzed. For most purposes, however, intervals $\Delta x = 5^\circ$ give fairly accurate results.

In an effort to save the time and labor required for determining the amplitudes A_n and B_n in a Fourier's series, many an inventor has been led to contrive mechanical or electro-mechanical means of evaluating these constants. It is outside the scope of this text to describe any of these "harmonic analyzers." The principle underlying their operation is Eq. (5). Besides being generally inaccurate for many purposes such analyzers are prohibitively costly.

8. Other Forms of Fourier's Series. — The reader is apt to meet a Fourier's series expressed in one or the other of the following forms:

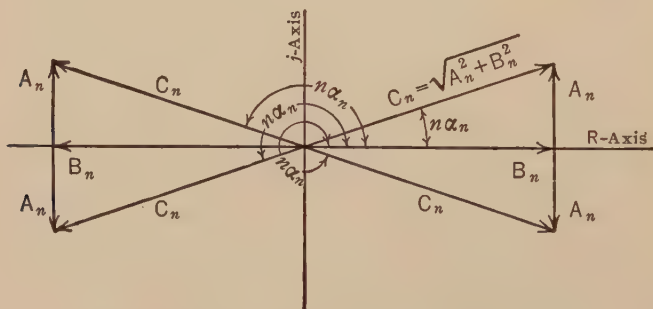
$$\left. \begin{aligned} f(x) &= 0.5 A_0 + \sum C_n \cos n(x + \beta_n) \quad (a) \\ f(x) &= 0.5 A_0 + \sum C_n \sin n(x + \alpha_n) \quad (b) \end{aligned} \right\} \quad (77)$$

It must not be inferred from this that the sine harmonics do not exist in (77a) and the cosine harmonics are lacking in (77b). That Eqs. (77) are identical with Eqs. (4) may be shown as follows:

TABLE II. THE INITIAL ANGLE $n\alpha_n$ AND THE PHASE OF C_n WITH RESPECT TO THE ORIGIN $x = 0$

Value of A_n	Value of B_n	Value of $n\alpha_n^*$	Phase of C_n with respect to the origin $x = 0^*$
Zero	Zero	$n\alpha_n = 0$	$C_n = 0$
Positive	Zero	$n\alpha_n = 90^\circ$	C_n leads the origin by 90°
Zero	Positive	$n\alpha_n = 0^\circ$	C_n is in phase with the origin
Zero	Negative	$n\alpha_n = 180^\circ$	C_n lags } origin by 180° leads }
Negative	Zero	$n\alpha_n = 270^\circ$	C_n lags origin by 90°
Positive	Positive	$0 < n\alpha_n < 90^\circ$	C_n leads origin by an angle $n\alpha_n$ less than 90°
Positive	Negative	$+90^\circ < n\alpha_n < 180^\circ$	C_n leads origin by an angle $n\alpha_n > 90^\circ$ but less than 180°
Negative	Positive	$-90^\circ < n\alpha_n < 0^\circ$	C_n lags origin by an angle greater than 0° but less than 90°
Negative	Negative	$180^\circ < n\alpha_n < 270^\circ$	C_n lags the origin by an angle greater than 90° but less than 180°

* Note. — The phase angles in columns 3 and 4 of this table are measured to the scale of the n th harmonic.

FIG 10. Vector diagram of the sine and cosine components of the n th harmonic and their sum.

Consider the sine and cosine terms of the n th harmonic in Eq. (4) and add them. This gives:

$$\left. \begin{aligned} A_n \cos nx + B_n \sin nx &= (\sqrt{A_n^2 + B_n^2}) \cos n(x + \beta_n) & (a) \\ &= C_n \cos n(x + \beta_n) & (b) \\ &= C_n \sin n(x + \alpha_n) & (c) \end{aligned} \right\} \quad (78)$$

where

$$\left. \begin{aligned} C_n &= \sqrt{A_n^2 + B_n^2} & (a) \\ n\alpha_n &= \tan^{-1} A_n/B_n & (b) \\ n\beta_n &= \tan^{-1} B_n/A_n & (c) \end{aligned} \right\} \quad (79)$$

Hence to express a Fourier's series in the form of Eqs. (77a) or (77b), select C_n so as to satisfy Eq. (79a) and $n\alpha_n$ and $n\beta_n$ so as to satisfy Eqs. (79b) and (79c) respectively. Since A_n and B_n may be positive, negative, or zero, the phase angles $n\alpha_n$ and $n\beta_n$ may range from 0 to 360°. The relation expressed by Eqs. (79a) and (79b) are shown graphically in Fig. 10, from which Table II may be derived.

9. Average Value of a Periodic Function. — The General Expression for the average value of any periodic function is given in Eq. (5), Chapter III, and is:

$$E_a = \frac{1}{\theta - \beta} \int_{\beta}^{\theta} f(x) dx \quad (80)$$

This integral (Fig. 11) represents the shaded area (bounded by the curve, the axis of abscissa, and the two ordinates erected at the points $x = \beta$ and $x = \theta$) divided by the base $(\theta - \beta)$.

Two methods of evaluating E_a suggest themselves:

- (a) Graphical method.
- (b) Analytical method.

(a) *The Graphical Method* consists of determining the area, referred to above, either by a planimeter or by some other means and dividing this area by the length of abscissa $(\theta - \beta)$. This letter must not be

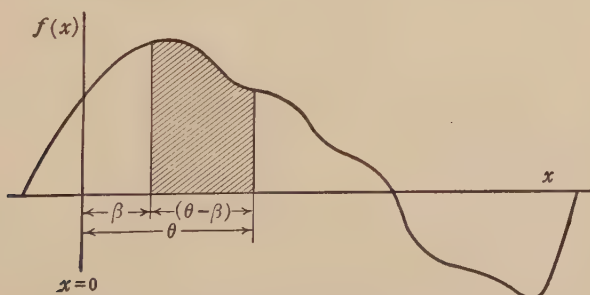


FIG. 11. Average value of a periodic function.

measured in degrees or radians but in centimeters or inches depending upon whether the area is determined in square centimeters or square inches respectively. This method possesses the advantages of being

simple, practical and fairly accurate. Its use in connection with functions that are not explicitly defined but given in the form of curves or oscillograms is recommended.

(b) *The Analytical Method* consists in substituting the value $f(x)$ in Eq. (80) and performing the required integration, then dividing the result by $(\theta - \beta)$ expressed in radians. Naturally this method is applicable to both explicitly and graphically defined functions provided that the latter have been expressed by a Fourier's series. We shall illustrate this method through two problems.

Problem 1. Determine the average value between $x = 0$ and $x = \pi$ of the function represented by Fig. (5) and explicitly defined by Eqs. (50).

Solution (a). — Since the function assumes different values in the interval $0 \leq x \leq \pi$ it will be necessary to perform the integration over two ranges. Substituting the values $f(x)$ from Eqs. (50) in Eq. (80) we have:

$$E_a = (1/\pi) \left[\int_0^{0.5\pi} ax \, dx + \int_{0.5\pi}^{\pi} a(\pi - x) \, dx \right] = a\pi/4 \quad (81)$$

This result could have been predicted because the area of the triangle Fig. 5 is $a\pi^2/4$. This divided by the base π gives $a\pi/4$.

Solution (b). — Substituting the Fourier's series representing the function (see Eq. 53) in Eq. (80) we obtain:

$$\left. \begin{aligned} E_a &= (4a/\pi^2) \int_0^{\pi} \left[(1/1^2) \sin x - (1/3^2) \sin 3x \right. \\ &\quad \left. + (1/5^2) \sin 5x - (1/7^2) \sin 7x \cdots + \right] dx \quad (a) \\ &= (4a/\pi^2) [2 - (2/9) + (2/25) - (2/49) + \cdots] = a\pi/4 \quad (b) \end{aligned} \right\} \quad (82)^*$$

Problem 2. — Evaluate the average value between 0 and 2π of the rectified sine wave represented by Fig. 6 and defined through Eq. (54).

Solution (a). — Substituting Eq. (54) in Eq. (80) we have:

$$E_a = (1/2\pi) \int_0^{2\pi} E_m \sin(x/2) \, dx = 2E_m/\pi \quad (83)$$

Solution (b). — Substituting the Fourier's series representing this function (see Eq. 57) in Eq. (80) we have:

$$\begin{aligned} E_a &= (4E_m/2\pi^2) \int_0^{2\pi} [0.5 - (1/3) \cos x - (1/15) \cos 2x \\ &\quad - (1/35) \cos 3x - \cdots] \, dx = 2E_m/\pi \end{aligned} \quad (84)$$

Although these problems are taken from explicitly defined functions they illustrate the general procedure of finding the average value when a function is defined by a Fourier's series.

*The *exact* average value of a periodic function cannot be found when the function is expressed by an infinite convergent series. The degree of accuracy depends upon the number of terms taken. For most practical purposes only the first four or five terms of the series need be considered.

10. Effective Value of a Periodic Function. — The General Expression for the effective value E of a periodic function is given in Eq. (8) Chapter III. This is:

$$E^2 = \frac{1}{\theta - \beta} \int_{\beta}^{\theta} [f(x)]^2 dx \quad (85)$$

Generally the limits of integration are taken as 0 and 2π . Hence, for the special case where $\beta = 0$ and $\theta = 2\pi$, Eq. (85) becomes:

$$E^2 = (1/2\pi) \int_0^{2\pi} [f(x)]^2 dx \quad (86)$$

Two methods of evaluating the effective value of a periodic function suggest themselves — the one graphical and the other analytical.

(a) *Graphical Method.* — The graphical method (see Fig. 12) consists in (1) squaring as many ordinates $f(x)$ of a given curve as the accuracy demands, (2) joining these squared ordinates by a smooth

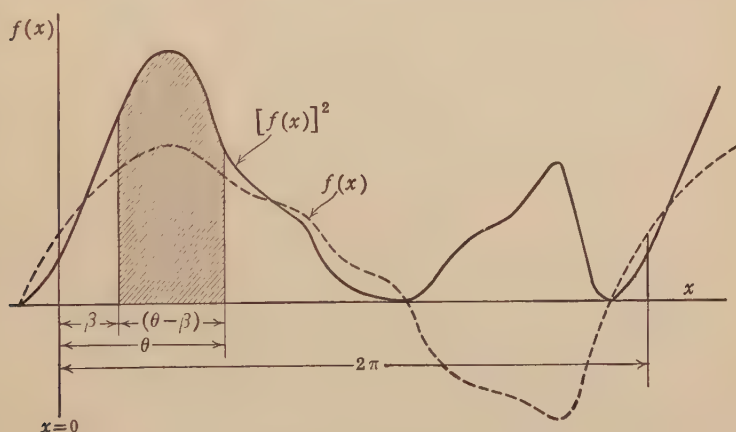


FIG. 12. Effective value of a periodic function.

curve, and (3) finding the shaded area (Fig. 12) of this curve by means of a planimeter. This area is then divided by the length of the base $(\theta - \beta)$ expressed in inches or centimeters. The square root of the quotient, so obtained, gives the effective value of the wave. Naturally this method leads to various inaccuracies. It is suggested as an easy and direct means of obtaining the effective value of graphically defined functions when the range of integration is other than 2π .

(b) *Analytical Method.* — This method consists in substituting the value $f(x)$ in Eq. (85) or (86) and performing the required integration,

then dividing the result by $(\theta - \beta)$ or by 2π as the case may be. We shall discuss this method under two cases:

- (1) The interval $\theta - \beta = 2\pi$
- (2) The interval $\theta - \beta \neq 2\pi$

Case (1). The Interval $\theta - \beta = 2\pi$. — We shall prove that, in the special case, when the integration is carried over a complete cycle (see Eq. 86) the effective value of a periodic function is:

$$E^2 = (0.5 A_0)^2 + 0.5 \sum C_n^2 \quad (87)$$

where C_n^2 is defined by Eq. (79a).

PROOF. — Let the general Fourier's series be expressed by Eq. (77b). Substituting this value in Eq. (86) we have:

$$E^2 = (1/2\pi) \int_0^{2\pi} \left[0.5 A_0 + \sum C_n \sin n(x + \alpha_n) \right]^2 dx \quad (88)$$

or

$$E^2 = (1/2\pi) \int_0^{2\pi} \left[(0.5 A_0)^2 + A_0 \sum C_n \sin n(x + \alpha_n) + \left\{ \sum C_n \sin n(x + \alpha_n) \right\}^2 \right] dx \quad (89)$$

But if s and n are integers and if $s \neq n$ then:

$$\begin{aligned} \left\{ \sum C_n \sin n(x + \alpha_n) \right\}^2 &= \sum C_n^2 \sin^2 n(x + \alpha_n) \\ &+ 2 \sum C_s C_n \sin s(x + \alpha_s) \sin n(x + \alpha_n) \end{aligned} \quad (90)$$

Hence:

$$\begin{aligned} E^2 &= (1/2\pi) \int_0^{2\pi} \left[(0.5 A_0)^2 + A_0 \sum C_n \sin n(x + \alpha_n) \right. \\ &\quad \left. + \sum C_n^2 \sin^2 n(x + \alpha_n) + 2 \sum C_s C_n \sin s(x + \alpha_s) \sin n(x + \alpha_n) \right] dx \end{aligned} \quad (91)$$

Now the second and last terms in this integral vanish.* Hence the integral reduces to the form:

$$E^2 = (1/2\pi) \int_0^{2\pi} \left[(0.5 A_0)^2 + \sum C_n^2 \sin^2 n(x + \alpha_n) \right] dx \quad (92)$$

which upon integration gives:

$$E^2 = (0.5 A_0)^2 + 0.5 \sum C_n^2 \quad \text{Q.E.D.} \quad (93)$$

or since $C_n^2 = A_n^2 + B_n^2$ (see Eq. 79a) we have:

$$E^2 = (0.5 A_0)^2 + 0.5 \sum (A_n^2 + B_n^2) \quad (94)$$

* The second term in Eq. (91) vanishes because the integral of a sine function over a complete cycle is zero. The last term vanishes by Theorem II, Appendix II.

Eqs. (93) and (94) state that the square of the effective value of a periodic wave (taken over a period) is equal to the square of the constant term plus one half the sum of the squares of the amplitudes of the harmonics. Moreover, since $E = (E_m/\sqrt{2})$, = $(C/\sqrt{2})$ Eq. (93) may be written as follows:

$$E^2 = E_0^2 + \sum E_n^2 \quad (95)$$

or the square of the effective value of a periodic wave (taken over a period) is equal to the square of the constant term plus the sum of the squares of the effective values of the harmonics.

Case (2). The Interval $\theta - \beta \neq 2\pi$. — We shall discuss this case under two headings; illustrating the method in each case by a specific problem.

(CASE 2A) *The function is explicitly defined not by a Fourier's series.*

(CASE 2B) *The function is explicitly defined by a Fourier's series.*

CASE (2A). — Let it be required to determine the effective value over the interval $\pi/4$ to $3\pi/4$ of the periodic function given in Fig. 5 and defined by Eqs. (50).

Solution. — Substituting the values of $f(x)$ as defined by Eq. (50) in Eq. (85) and integrating over two ranges we have:

$$\begin{aligned} E^2 &= \frac{1}{(3\pi/4) - (\pi/4)} \left[\int_{\pi/4}^{\pi/2} (ax)^2 dx + \int_{\pi/2}^{3\pi/4} (a\pi - ax)^2 dx \right] \\ &= 7 a^2 \pi^2 / 48 \end{aligned} \quad (96)$$

whence $E = 0.382 a\pi = 1.2 a$

CASE (2B). — Let it be required to evaluate the effective value over the interval $\pi/4$ to $3\pi/4$ of the periodic function given in Fig. 5 and defined by Fourier's series Eq. (53).

Solution. — Substituting Eq. (53) in Eq. (85) we have:

$$\begin{aligned} E^2 &= \frac{1}{(3\pi/4) - (\pi/4)} \int_{\pi/4}^{3\pi/4} \left[(4a/\pi) \{ \sin x - (1/3^2) \sin 3x \right. \\ &\quad \left. + (1/5^2) \sin 5x - (1/7^2) \sin 7x \cdot \cdot \cdot \} \right]^2 dx \end{aligned} \quad (97)$$

squaring the Fourier's series, integrating and substituting limits we have:

$$E = 0.382 a\pi = 1.2 a$$

It must be here emphasized that when a function is defined through an infinite series the process of determining the exact effective value borders on the realm of impossibility because theoretically an infinite number of men will have to spend their life time evaluating, for example,

Eq. (97). Since the series is convergent only the first two or three terms need be taken and the result would be accurate enough for most practical purposes. Let us investigate how accurate the result would be if only the first two terms of the series are taken.

Eliminating all harmonics above the third, Eq. (97) reduces to:

$$E^2 \cong (2/\pi) \int_{0.25\pi}^{0.75\pi} (4a/\pi)^2 \left[\sin^2 x + (1/81) \sin^2 3x - (2/9) \sin x \sin 3x \right] dx \quad (98a)$$

Using the trigonometric transformation $\sin a \sin b = [\cos(a-b) - \cos(a+b)]/2$ we obtain:

$$E^2 = (2/\pi) \int_{0.25\pi}^{0.75\pi} (4a/\pi)^2 \left[(83/162) - (5/9) \cos 2x + (1/9) \cos 4x - (1/162) \cos 6x \right] dx \quad (98b)$$

Integrating we have:

$$E^2 = (32a^2/\pi^3) \left[(83/164)x - (5/18) \sin 2x + (1/36) \sin 4x - (1/972) \sin 6x \right]_{0.25\pi}^{0.75\pi} = 1.18a \quad (98c)$$

which is close enough to $(1.2a)$ for most practical purposes.

11. Form and Amplitude Factors. — The definitions of the amplitude and form factors given in Eqs. (11) and (12), Chapter III, apply to all periodic functions. These are:

$$\left. \begin{aligned} k_a &= E_m/E & (a) \\ k_f &= E/E_a & (b) \end{aligned} \right\} \quad (99)$$

where

k_a = amplitude factor.

k_f = form factor.

E_m = maximum value of the function $f(x)$.

E = effective value of the function.

E_a = average value of the function.

The values E and E_a may or may not be taken over an interval of 2π but they must be computed for the identical interval $\theta - \beta$. Engineering practice, however, favors taking the interval as 2π in both cases.

12. The Concept of Phase in Non-Sine Waves and the Representation of Non-Sine Functions by Revolving Vectors. — Let a non-sine wave consist of a sine fundamental and a sine second harmonic. In other words, let

$$e = f(t) = E_{m1} \sin \omega t + E_{m2} \sin 2\omega t \quad (100)$$

The two sine harmonics and the non-sine wave defined by Eq. (100) are represented graphically by Fig. 13a. The rotating vectors $\mathbf{E}_1$ and $\mathbf{E}_2$, which generate the sine waves e_1 and e_2 , are shown in Fig. 13b. The angular velocity of the fundamental is ω , whereas that of the second harmonic is 2ω . Consequently the vector generating the second harmonic completes two revolutions for each revolution of the fundamental. In other words, in the same interval t , the revolving vector $\mathbf{E}_1$ sweeps through an angle θ which is half as large as that swept over by the revolving vector $\mathbf{E}_2$.

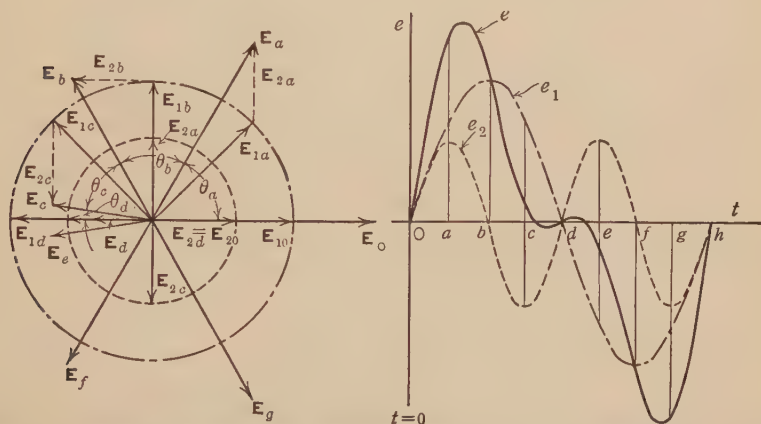


FIG. 13b. Vector diagram of the non-sine wave Fig. 13a.

FIG. 13a. A non-sine wave consisting of a fundamental and a second sine harmonic.

Now consider various instants of time $t_0, t_a, t_b, \dots, t_h$. At the instant t_0 the two vectors $\mathbf{E}_{10}$ and $\mathbf{E}_{20}$ are in phase and their sum $\mathbf{E}_0$ represents the vector which generates the non-sine wave. At the instant t_a the vectors $\mathbf{E}_{1a}$ and $\mathbf{E}_{2a}$ will have rotated through 45° and 90° respectively and their vector sum $\mathbf{E}_a$ makes an angle θ_a that is greater than 45° but less than 90° . This process (see Fig. 13b) is repeated through each step, that is, for every angle θ that the fundamental traverses the second harmonic moves through an angle 2θ . The results are:

(a) The phase angle between the two vectors is not constant at every instant. Thus for t_0 the phase angle between $\mathbf{E}_{10}$ and $\mathbf{E}_{20}$ is zero; for t_a the phase angle between $\mathbf{E}_{1a}$ and $\mathbf{E}_{2a}$ is 45° ; for t_b it is 90° , etc.

(b) The vector sum of the two harmonics, which is the vector generating the non-sine wave, possesses the following peculiarities:

(1) It is not constant in magnitude as is demonstrated by Fig. 13b where:

$$E_0 \neq E_a \neq E_b \neq E_c \quad (101)$$

(2) For equal increments of time t , the vector $\mathbf{E}$ does not revolve through equal angles. Thus, although in Fig. 13a equal increments of time are taken between the points 0 and a , a and b , b and c , etc., the angles $\theta_a, \theta_b, \dots \theta_h$, between successive positions of the vector $\mathbf{E}$, Fig. 13b, are not equal.

Now the revolving vectors, which generate sine waves, have constant magnitudes and cover equal angles for equal increments of time. Although, therefore, a non-sine wave may be represented by a revolving vector, this vector differs fundamentally from the vector representing a sine wave. The graphic representation of non-sine waves by rotating vectors is a very laborious process. Moreover, such vectors are neither conducive to a clearer understanding of the underlying physical phenomena, nor do they facilitate the mathematical treatment. We shall therefore discard them in the treatment of non-sine waves. Again, since coplanar vectors and complex numbers are inseparable — the one for a graphic representation of the physical facts and the other for the analytical treatment thereof, we shall also discard the use of complex numbers in developing the current and voltage relations for non-sine waves and shall limit the discussion of the succeeding chapter to the instantaneous and effective value relations.

The use of complex numbers and vectors in circuit analysis has become a fixed habit in engineering practice. Hence, even though the Emf. and current are not sine functions, the tendency is to represent the non-sine function by the so-called "equivalent sine function," that is, a sine function which produces the same effective value relations as the non-sine function. This practice is dangerous and is apt to lead to inaccurate and even absurd results unless the problem is handled carefully and competently. We recommend the use of the methods outlined in the succeeding chapter before any approximations are made. Then the limit of these approximations will be determined when a comparison is made with the actual accurate results.

PROBLEMS

1a. It is shown in Chapter III that when a sine wave is a function of time, the angle x in Eq. (1) may be replaced by ωt where $\omega = 2\pi f$.

(a) Express Eqs. (2) as functions of time.

(b) Show that the revolving vectors generating the first, second, and third harmonics have angular velocities ω , 2ω , and 3ω .

1b. Plot on graph paper the two sine functions:

$$f(x) = A \sin(x + 30^\circ)$$

$$\phi(x) = B \sin 1.5(x + 30^\circ)$$

Indicate on the graph (a) the origin $x = 0$, (b) the amplitudes A and B , and (c) the scales to which the curves are plotted.

1c. Plot the graph of the function which is the sum of the two sine functions given in problem 1b.

2. Draw the graph of a periodic non-sine function for two cycles, and indicate on the graph at least 8 points any two of which are a period (2π) apart. Does each couple satisfy the condition for periodicity, namely, $f(x) \equiv f(\pi + x)$?

3a. Define a function which does not lend itself to a representation by a Fourier's series within certain intervals:

- (a) Because it does not satisfy the continuity condition.
- (b) Because it is not finite.
- (c) Because it is not single-valued.

3b. Draw the graph of the function defined in problem 3a for the three intervals where:

- (a) The function possesses a relatively large number of discontinuities.
- (b) The function is infinite.
- (c) The function is multi-valued.

4a. A periodic function (in 2π) satisfies the symmetry condition $f(x) \equiv -f(2\pi - x)$. Show that this function also satisfies the symmetry $f(x) \equiv -f(-x)$.

Solution. — The identity $f(x) \equiv -f(2\pi - x)$ must hold true for all values of x and consequently for the particular value $x_1 = (2\pi + x)$. Substituting this value in the identity we have:

$$f(2\pi + x) \equiv -f(2\pi - 2\pi - x) \equiv -f(-x) \quad (a)$$

But by the definition of periodicity $f(2\pi + x) \equiv f(x)$. Hence Eq. (a) becomes:

$$f(x) \equiv -f(-x) \quad \text{Q.E.D.}$$

4b. Show that if a periodic function satisfies the symmetry $f(x) \equiv f(2\pi - x)$ then it also satisfies the symmetry $f(x) \equiv f(-x)$.

4c. Show that if a periodic function satisfies the symmetries:

$$\begin{aligned} f(x) &\equiv -f(-x) \\ f(x) &\equiv f(\pi - x) \end{aligned}$$

then it also satisfies the following symmetries:

$$\begin{aligned} f(x) &\equiv f(2\pi - x) \\ f(0.5\pi + x) &\equiv f(0.5\pi - x) \\ f(x) &\equiv -f(\pi + x) \end{aligned}$$

4d. Draw a non-sine periodic function satisfying all the symmetries given in problem 4c. Does a sine wave satisfy these symmetries?

4e. Show that if a periodic function satisfies the symmetries

$$\begin{aligned} f(x) &\equiv -f(\pi + x) \\ f(x) &\equiv (\pi - x) \end{aligned}$$

then it also satisfies the symmetries:

$$\begin{aligned} f(x) &\equiv -f(-x) \\ f(x) &\equiv f(2\pi - x) \\ f(0.5\pi + x) &\equiv f(0.5\pi - x) \end{aligned}$$

4f. In Table I, there are six independent types of symmetry. Make a list comprising various combinations of any two of them and determine which of the remaining symmetries follow.

4g. Show that if a function satisfies the symmetry condition $f(x) \equiv f(\pi - x)$, then it also satisfies the symmetry condition $f(x) \equiv f(-x)$ provided that a new origin $x = 0$ is selected at the point $x = \pm \pi/2$.

(Hint. — Examine Fig. 4f and convince yourself that this is, at least, plausible, then proceed with the proof.)

5a. A periodic function satisfies the symmetry $f(x) \equiv f(2\pi - x)$ in the interval $0 \leq x \leq 2\pi$.

(a) Determine what harmonics are eliminated due to this symmetry

(b) Give the Fourier's series representing this function.

5b. A periodic function satisfies the symmetries

$$f(x) \equiv -f(-x)$$

$$f(x) \equiv -f(\pi - x)$$

(a) Sketch the graph of the function showing the origin $x = 0$.

(b) Prove that, due to these symmetries, the Fourier's series representing this function reduces to the form:

$$f(x) = \sum B_q \sin qx \quad (q \text{ an even integer})$$

5c. A periodic function satisfies the symmetries

$$f(x) \equiv -f(\pi + x)$$

$$f(x) \equiv -f(\pi - x)$$

(a) Sketch the graph of the function showing the origin $x = 0$.

(b) Prove that due to these symmetries the Fourier's series for this function reduces to the form:

$$f(x) = \sum A_p \cos px \quad (p \text{ an odd integer})$$

5d. Problems 5b and 5c are taken from Table I by combining symmetries (a), (c) and (b), (c) respectively. Many more similar problems could be designed by taking various combinations of any two of the symmetries. Referring to the list already made in problem 4f:

(a) Write the Fourier's series representing the functions which simultaneously satisfy any two symmetries.

(b) In order to visualize these functions, draw graphs satisfying these symmetries showing the origin $x = 0$.

6a. Show that symmetries (c) and (f) do not respectively result in any simplification of the integral Eq. (5').

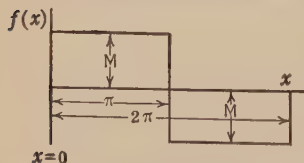


FIG. 14. Mmf. of a distributed winding.

Each half-wave is a rectangle of height M .

6b. In Art. 6f the effect of symmetries (b) and (f) on the integral Eq. (5') is shown. Referring to the list in problem 4f, show the effect of the various symmetry combinations, given in that list, on the integral Eq. (5').

7a. Fig. 14 represents approximately the Mmf. wave of a distributed winding in a non-salient

- (a) Define the function for the interval $0 \leq x \leq \pi/2$.
- (b) Examine the function for symmetries and state the symmetry conditions which it satisfies, making sure to specify the origin $x = 0$.
- (c) Give the Fourier's series which defines this function.
- (d) Evaluate the constants of the series.

7b. The graph (Fig. 15) represents the distribution over the rotor, of the exciting turns of a two-pole turbo-alternator.

- (a) Define the function represented by this graph for the interval $0 \leq x \leq \pi/2$.
- (b) Examine the function for symmetries and state the symmetry conditions which it satisfies, making sure to specify the origin $x = 0$.
- (c) Give the Fourier's series which represents this function.
- (d) Evaluate the constants of the series.

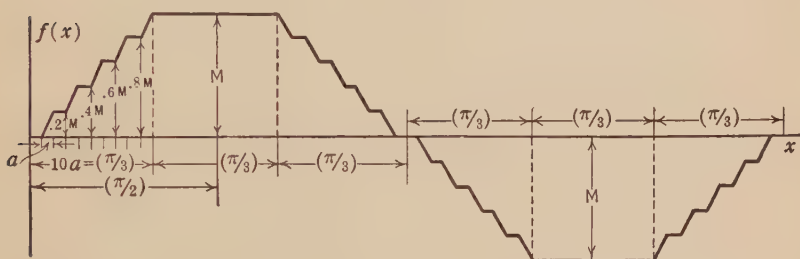


FIG. 15. Mmf. of a turbo-alternator rotor.

7c. In order to minimize the effect of pulsation two rectifiers, drawing power from a two-phase alternator, are connected in series. The resultant rectified waves are shown in Fig. 16. If the origin is taken as shown, then the total Emf. across the terminals of the rectifier-set may be defined as follows:

$$f(t) = e = E_m [\sin (\omega t/2) + \cos (\omega t/2)] \quad 0 \leq t \leq (T/2)$$

$$f(t) = e = E_m [\sin (\omega t/2) - \cos (\omega t/2)] \quad (T/2) \leq t \leq T$$

- (a) What symmetry condition does this function satisfy?
- (b) Define the function by a Fourier's series.
- (c) Evaluate the constants of the series.

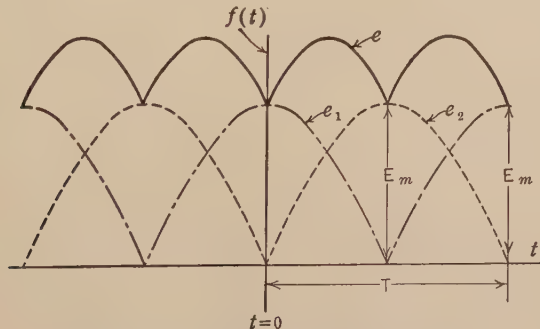


FIG. 16. A rectified two-phase wave.

7d. The rectified current of a certain type of rectifier consists of only half a sine wave, the other half being completely cut off (see Fig. 17). The equation for the rectified current wave is:

$f(x) = 0$ for the interval $0 \leq x \leq \pi/2$
 $f(x) = -I_m \cos x$ for the interval $(\pi/2) \leq x \leq (3\pi/2)$
 $f(x) = 0$ for the interval $(3\pi/2) \leq x \leq 2\pi$

- (a) What symmetry conditions does the function satisfy?
- (b) Define the function by a Fourier's series using ωt instead of x , as the variable angle.
- (c) Evaluate the constants of the series.

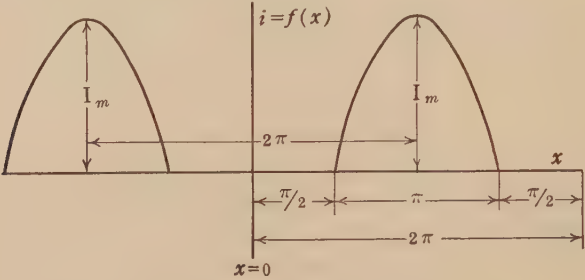


FIG. 17. A rectified current.

7e. The ordinates of the curve representing flux density $\mathfrak{B}$ in the air-gap of a loaded continuous-current generator are tabulated as follows:

Elec. Deg.	0	5	10	15	20	25	30	35	40	45	50
$\mathfrak{B}/\mathfrak{B}_m$	0.0	0.3	0.6	0.8	0.95	1	0.99	0.92	0.86	0.82	0.79

Elec. Deg.	55 to 140			145	150	155	160	165	170	175	180
$\mathfrak{B}/\mathfrak{B}_m$	0.78			0.7	0.66	0.61	0.56	0.46	0.38	0.24	0.0

The curve is represented by Fig. 4b and satisfies the symmetry $f(x) \equiv -f(\pi + x)$.

- (a) Define this curve by a Fourier's series.
- (b) Evaluate the constants of the series.

7f. If Fourier's series is true at all it must be true of sine and cosine functions. Show that the Fourier's series, for such functions, reduces to an identity.

7g. For some purposes it is convenient to represent the saturation curve of a dynamo by a Fourier's series. This curve is generally plotted with exciting amperes as abscissa against induced Emf. as ordinates. Call 2π the length of the abscissa over which the curve is plotted. Divide this length into a convenient number of parts. Measure ordinates corresponding to each point of division and construct a table similar to that given in problem 7e. The series representing such a curve is given by Eq. (4) and the constants A_n and B_n in the series may be evaluated by the use of Eqs. (74). For evaluating the constant term $0.5 A_0$ see operation (13), Art. 7. Select such a curve and express the functional relation, between the flux density $\mathfrak{B}$ and the magnetic intensity $\mathcal{H}$, by a Fourier's series.

8a. Express the Fourier's series obtained in problem 7d as a sum of sine harmonics having phase angles α_n .

8b. Express the Fourier's series obtained in problem 7d as a sum of cosine harmonics displaced from the origin by angles β_n .

9a. Find the average value (in the interval 0 to $\pi/2$) of the function given in problem 7b.

9b. Determine the average value in the interval 0 to 2π of the wave (Fig. 16):

(a) By using the value of the function as defined in problem 7c.

(b) By using the Fourier's series, and taking only the first four terms.

(c) Graphically.

9c. Plot the curve given in problem 7d and determine its average value graphically over the range 0 to π .

9d. Compute the average value of the wave (problem 7d) by substituting for $f(x)$ in Eq. (80) its value in terms of the Fourier's series and check the result so obtained with the result of problem 9c.

Note.—Take only the first four terms of the series.

10a. Determine the effective value in the interval 0 to $\pi/2$ of the function given in problem 7b and show that this is approximately equal to the effective value as determined from Eq. (95).

Note.—Use the first six terms of the series.

10b. Determine the effective value from 0 to 2π of the rectified sine wave (Fig. 16):

(a) By using the value of the function as defined in problem 7c.

(b) By using Eq. (95).

10c. Determine the effective value over the range 0 to $\pi/2$ of the rectified sine wave (Fig. 16):

(a) By using the value of the function as defined in problem 7c.

(b) By using the first two terms of the Fourier's series representing this function.

(c) Compare the approximate value obtained in (b) with the actual value as obtained in (a). What is the per cent of error?

10d. Compute the effective value in the interval 0 to 2π of the wave given in problem 7d.

11. Determine the form factors of the waves given in problems 7a, 7b, 7c and 7d.

12. Study Figs. 13 carefully; then draw similar figures for a non-sine wave consisting of a fundamental and a third harmonic.

CHAPTER XIV

NON-SINE PERIODIC EMFS. AND CURRENTS

1. Introduction. — Non-sine periodic waves are of more frequent occurrence in electrical engineering practice than sine waves. Although efforts are expended in order to generate sine voltages, the Emfs. actually obtained in electrical machines do deviate from the sine form. The analysis of sound waves, which are transmitted electrically over telephone or radio, shows that such waves are non-sine waves. Hence it is essential to study the electrical relations in circuits subjected to non-sine Emfs.

Fortunately, such waves (see Chapter XIII) can be resolved into a sum of sine waves, of various frequencies, known as harmonics. This greatly facilitates the problem, which otherwise would be insurmountable. Indeed, the electrical profession is greatly indebted to the French physicist Fourier who, in the course of his analytical studies of heat conduction, discovered the fact that a periodic non-sine function (such as that shown in Fig. 1) may be expressed by the sum of a constant E_0 and several sine functions having amplitudes $E_{m1}, E_{m2} \dots E_{mn}$ frequencies, $f_1, 2 f_1, 3 f_1 \dots n f_1$, and initial angles $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$.

The sine function of amplitude E_{m1} , frequency f_1 , and initial angle α_1 is called the fundamental or first harmonic (see Fig. 1). Its instantaneous value equation is:

$$e_1 = E_{m1} \sin (\omega_1 t + \alpha_1) \quad (1)$$

In general, the sine function whose amplitude is E_{mn} , frequency $n f_1$, and initial angle α_n is known as the n th harmonic. Its instantaneous value expression is:

$$e_n = E_{mn} \sin n(\omega_1 t + \alpha_n) \quad (2)$$

In order to visualize these facts consider the non-sine curve, Fig. 1. Any ordinate e corresponding to an abscissa $\omega_1 t$ represents the algebraic sum of the corresponding ordinates e_1, e_3, e_4 , and e_5 of the fundamental, third, fourth, and fifth harmonics, respectively. In general:

$$e = f(t) = E_0 + e_1 + e_2 + e_3 + \dots + e_n \quad (3)$$

where:

$$\left. \begin{array}{ll} E_0 = \text{a constant term} & (a) \\ e_1 = E_{m1} \sin (\omega_1 t + \alpha_1) & (b) \\ e_2 = E_{m2} \sin 2 (\omega_1 t + \alpha_2) & (c) \\ \cdot & \cdot \\ \cdot & \cdot \\ e_n = E_{mn} \sin n(\omega_1 t + \alpha_n) & (n) \end{array} \right\} \quad (4)$$

Substituting Eqs. (4) in (3) we have:

$$\left. \begin{aligned} e &= E_0 + E_{m1} \sin(\omega_1 t + \alpha_1) + E_{m2} \sin 2(\omega_1 t + \alpha_2) \\ &\quad + E_{m3} \sin 3(\omega_1 t + \alpha_3) + \cdots + E_{mn} \sin n(\omega_1 t + \alpha_n) \quad (a) \\ &= E_0 + \sum_{n=1}^{\infty} E_{mn} \sin n(\omega_1 t + \alpha_n)^* \quad (b) \end{aligned} \right\} (5)^*$$

Eqs. (5) are identical expressions for the so-called *Fourier's series*. They state that a periodic function $f(t)$ may sometimes be represented by a series consisting of a constant term and a sum of sine terms whose frequencies are integer multiples of the frequency f_1 of the fundamental.

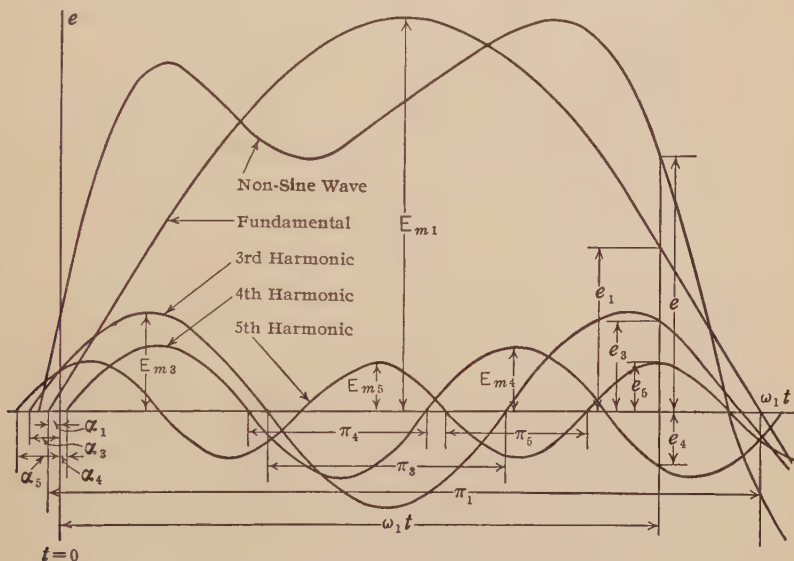


FIG. 1. A non-sine wave and its harmonics.

* The average student is generally afraid to use the summation sign, $\sum$. This fear probably arises from the fact that he does not understand its meaning. Let

us state that the expression $\sum_{n=1}^{\infty} C_n$ implies the following operation:

- Use 1 as subscript and obtain the term C_1 .
- Place a plus sign after C_1 and use 2 as subscript and obtain $C_1 + C_2$.
- Place a plus sign after C_2 and use 3 as subscript thereby obtaining $C_1 + C_2 + C_3$.
- Continue this process indefinitely and you have

$$\sum_{n=1}^{\infty} C_n = C_1 + C_2 + C_3 + \cdots + C_n + \cdots$$

The student can easily see that, if the summations in Eq. (4b) are expanded in this manner, the two expressions (4a) and (4b) would be identical.

Reference to Fig. 1 and Eqs. (5) shows that the fundamental has the same period as the non-sine wave and that the various harmonics have frequencies which are integer multiples of the fundamental. The question now arises as to how the abscissa scales of the various harmonics are related.

The n th harmonic, Fig. 1, makes n complete cycles for one cycle made by the fundamental. But each cycle consists of 360° . Therefore, for every 360° covered by the fundamental, the n th harmonic covers $n \times 360^\circ$; that is, every degree on the scale of the fundamental is equal to n degrees on the scale of the n th harmonic. This relationship of scales is accounted for by multiplying the *angles* (within parenthesis in Eqs. 4 and 5), *all of which are measured to the scale of the fundamental*, by the number of the harmonic.

In the following treatment it will be assumed that the instantaneous value of a non-sine periodic Emf. or current may be always represented by a Fourier's series (see Eqs. 5). We shall further assume that the effective value of such a non-sine wave may be computed from the following expression:*

$$E = [E_0^2 + E_1^2 + E_2^2 + E_3^2 + \dots + E_n^2]^{1/2} = (\sum E_n^2)^{1/2} \quad (6)$$

where E is the effective value of the non-sine wave and $E_1, E_2, E_3 \dots E_n$ are the effective values of the first, second, third $\dots$ n th harmonics respectively.

2. Circuits Consisting of a Resistance R and a Non-Sine Emf. e . —

(a) *Physical Considerations.* — Let a circuit (Fig. 2a) consist of a resistance R and a source of non-sine Emf. e . The voltage and current waves (Fig. 2b) are similar in every respect. In other words, the curves for e and i are such that if the deflections of the current and voltage elements of an oscillograph are properly adjusted the two waves would correspond point by point and there would be just one curve registered for both e and i . This constant proportionality of corresponding ordinates of the Emf. and current waves may be expressed as follows:

$$(e/i) = R \quad (7)$$

Next let these curves be analyzed into a Fourier's series (see Art. 7, Chapter XIII) and let the component harmonics be plotted as in Fig. 2c. The following facts are observed:

- (1) The same harmonics that are present in the voltage wave are also present in the current wave.
- (2) The phase angles of corresponding harmonics in the current and voltage waves measured from an arbitrary instant $t = 0$

* For a proof of Eq. (6) see Art. 10, Chapter XIII.

are identical; that is, if α_1 and α_3 are the initial angles of the first and third harmonics of the voltage wave, then these are also the initial angles of the corresponding current harmonics.

- (3) The amplitudes of corresponding harmonics of current and voltage bear a constant ratio.* Furthermore this ratio is identical with that defined by Eq. (7). In other words:

$$(E_{m1}/I_{m1}) = (E_{m3}/I_{m3}) = R \quad (8)$$

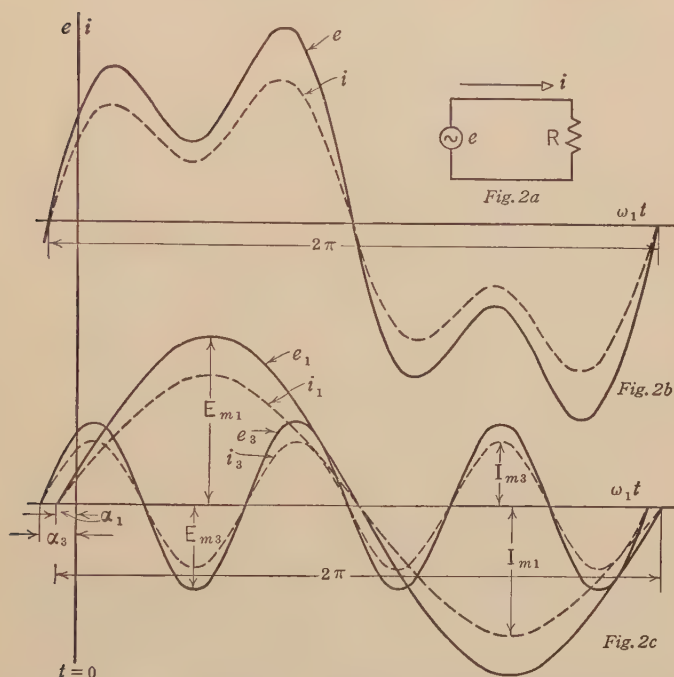


FIG. 2a. A circuit consisting of a resistance R and a non-sine Emf., e .

FIG. 2b. Emf. and current waves in the circuit Fig. 2a.

FIG. 2c. Harmonics obtained by analyzing the waves in Fig. 2b.

An ammeter and voltmeter connected in the circuit give effective values of the non-sine current and voltage which are so related that:

$$(E/I) = R \quad (9)$$

where R is simply a coefficient of proportionality between E and I and is the resistance of the circuit.

(b) *Mathematical Theory.* — These physical facts may be interpreted mathematically as follows:

* It is here assumed that the skin effect is negligible.

Let the instantaneous value of the current flowing through the circuit be expressed by the Fourier's series:

$$i = \sum I_{mn} \sin n(\omega_1 t + \beta_n) \quad (10)$$

By Ohm's Law, the instantaneous value of the voltage, which will cause this current to flow against the resistance of the circuit, is:

$$e = Ri \quad (11)$$

Substituting this value of i in Eq. (10) we have:

$$\left. \begin{aligned} e &= Ri = \sum RI_{mn} \sin n(\omega_1 t + \beta_n) & (a) \\ &= E_{mn} \sin n(\omega_1 t + \beta_n) & (b) \end{aligned} \right\} \quad (12)$$

$$\text{where} \quad E_{mn} = RI_{mn} \quad (13)$$

Or since the effective value of a sine wave is equal to $E_m/\sqrt{2}$ we have:

$$E_n = RI_n \quad (14)$$

We conclude, therefore, that when a non-sine voltage is applied to a resistance circuit, each harmonic acts as if it were alone obeying the laws developed for sine waves.

Comparing the Fourier's series of the voltage wave with that of the current wave, we note that the first three physical facts enumerated above hold true. It remains to show that the ratio of the effective value (E/I) is equal to R .

Using Eq. (6) for the effective value of a non-sine wave, we have:

$$(E/I) = \left[\sqrt{\sum (RI_n)^2} / \sqrt{\sum I_n^2} \right] = R \quad (15)$$

3. Circuits Consisting of an Inductance $\mathcal{L}$ and a Non-Sine Emf. e . —

(a) *Physical Considerations.* — Fig. 3a shows a circuit consisting of an inductance $\mathcal{L}$ and a source of non-sine Emf., e . The waves that would be recorded by a two-element oscillograph properly connected in this circuit are shown in Fig. 3b.

These waves are by no means similar. On the contrary, the current wave (i) appears to be, on the whole, "smoother" than the voltage wave (e). Unlike the case of a resistance circuit, there exists no constant coefficient of proportionality between corresponding ordinates of the voltage and current waves.

Upon analyzing these waves into Fourier's series (Fig. 3c) we find that:

- (1) The same harmonics that are present in the current wave are also present in the voltage wave.
- (2) The current harmonics lag the corresponding voltage harmonics by 90° measured to the scale of these harmonics.

- (3) The amplitudes of corresponding current and voltage harmonics do not bear a constant ratio. But the ratio decreases as the order of harmonic increases, with the result that:

$$(I_{m3}/E_{m3}) < (I_{m1}/E_{m1}) \quad (16)$$

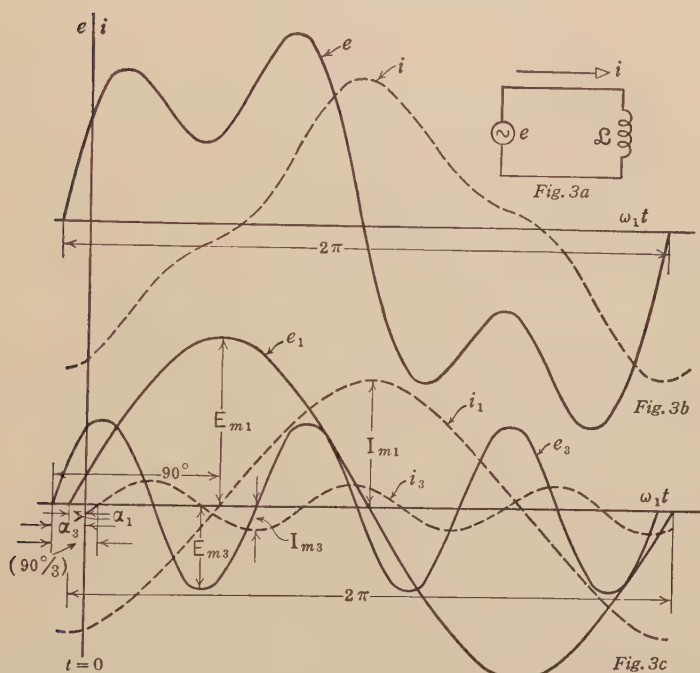


FIG. 3a. A circuit consisting of an inductance $\mathcal{L}$ and a non-sine Emf., e .

FIG. 3b. Emf. and current waves in the circuit Fig. 3a.

FIG. 3c. Harmonics obtained by analyzing the waves in Fig. 3b.

Now if an ammeter and a voltmeter are connected in the circuit, it will be found that the ratio of the effective values of the non-sine Emf. and current is not the same as the ratio of the effective value of any harmonic voltage and current; or:

$$(E/I) \neq (E_n/I_n) \quad (17)$$

These physical facts may be interpreted mathematically as follows.

(b) *Mathematical Theory.*—The voltage induced in an inductance due to a varying current is $e = -\mathcal{L} di/dt$, and the impressed voltage required to counterbalance this induced voltage is:

$$e = \mathcal{L} di/dt \quad (18)^*$$

* The skin effect is here again neglected.

Let the current i , in Eq. (10), be substituted in Eq. (18). This gives after differentiation:

$$\left. \begin{aligned} e &= \sum n\omega_1 \mathcal{E} I_{mn} \cos n(\omega_1 t + \beta_n) & (a) \\ &= \sum E_{mn} \sin n(\omega_1 t + \beta_n + 90^\circ/n) & (b) \end{aligned} \right\} \quad (19)$$

where:

$$E_{mn} = n\omega_1 \mathcal{E} I_{mn} = n(2\pi f_1 \mathcal{E}) I_{mn} = X_{Ln} I_{mn} = nX_{L1} I_{mn} \quad (20)$$

Dividing both sides of Eq. (20) by $\sqrt{2}$ and remembering that $E_{mn}/\sqrt{2} = E_n$ we have:

$$E_n = X_{Ln} I_n = nX_{L1} I_n \quad (21)$$

Again from Eq. (20) we have the following relations:

$$\left. \begin{aligned} X_{Ln} &= n(2\pi f_1 \mathcal{E}) & (a) \\ \text{Therefore: } B_{Ln} &= 1/X_{Ln} = 1/(n 2\pi f_1 \mathcal{E}) & (b) \end{aligned} \right\} \quad (22)$$

Comparing Eqs. (10) and (19) we find that the same harmonics which are present in the voltage wave do exist in the current wave and that the voltage harmonics lead the corresponding current harmonics by 90° measured to the scale of that harmonic.

Eq. (22) shows that *the inductive reactance of the circuit to the n th harmonic is n -times as great as its reactance to the fundamental*. Hence, for the same Emf., the higher the order of the harmonic the smaller would be the current. This is why the third harmonic current (Fig. 3c) is about one sixth the value of the fundamental current although the amplitude of the third harmonic voltage is only about one half that of the fundamental voltage. Inductance coils are used in practice to "smooth out" an irregular wave. Their action is dependent on this very fact, that their reactance increases with the order of the harmonic.

The question now arises as to what is the inductive reactance of the circuit, as a whole. The answer is that such a reactance has no physical existence whatever. We may, however, define an *equivalent inductive reactance* such that:

$$X_L = (E/I) = \left[\sum E_n^2 / \sum I_n^2 \right]^{1/2} = \left[\sum (X_n I_n)^2 / \sum I_n^2 \right]^{1/2} \quad (23)^*$$

* This definition of equivalent reactance arises from the tendency (in practice) to replace the non-sine waves of Emf. and current by equivalent sine waves whose effective values are equal respectively to the effective values of the non-sine Emf. and current. In the author's opinion this practice should be discouraged because it not only obscures the physical facts but is liable to give absurd results unless used strictly within its limitations. It must be here emphasized that the equivalent reactance, as defined by Eq. (23), holds true only for a given Emf. wave-form. This presupposes that certain harmonics exist in the Emf. wave and that the amplitudes of these harmonics bear certain definite relations to each other.

4. Circuits Consisting of a Capacitance C and a Non-Sine Emf. e .*

(a) *Physical Considerations.* — The circuit, Fig. 4a, consists of a capacitance C and a source of non-sine Emf. e . A properly connected oscillograph would register waves as shown in Fig. 4b. These waves are not similar. On the contrary, the irregularities of the Emf. wave are

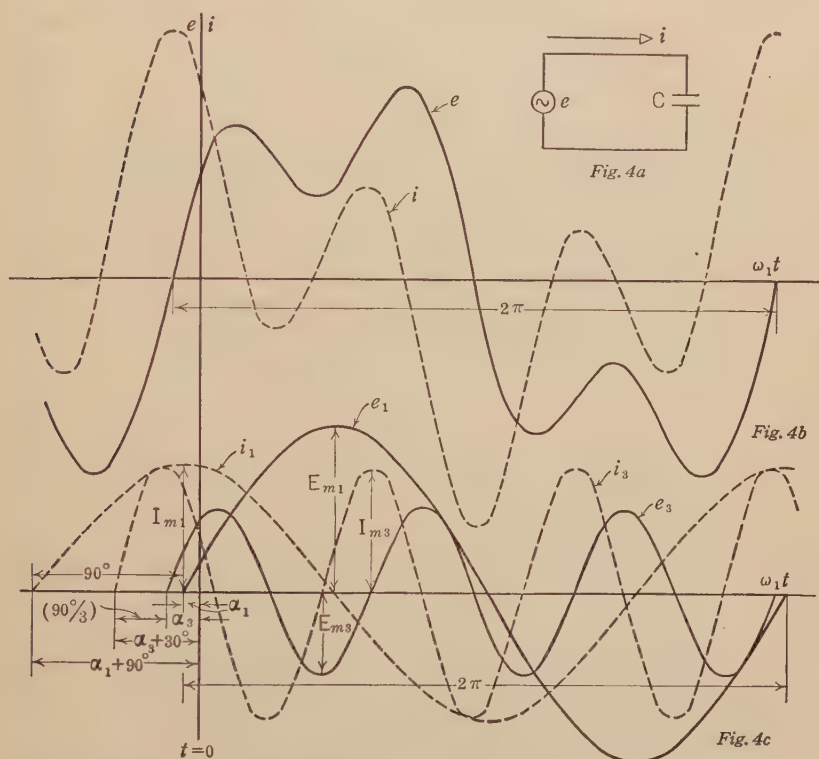


FIG. 4a. A circuit consisting of a capacitance C and a non-sine Emf., e .

FIG. 4b. Emf. and current waves in the circuit, Fig. 4a.

FIG. 4c. Harmonics obtained by analyzing the waves in Fig. 4b.

accentuated in the current wave. This effect is exactly the reverse of what happens in an inductance circuit where the current wave appears "smoother" than the Emf. wave.

If the waves are expressed by a Fourier's series and the component harmonics plotted as in Fig. 4c, the following facts would be learned:

* We assume here that the capacitance C is independent of the frequency. In other words, the dielectric medium is perfect and free from hysteresis, viscosity and all other dielectric anomalies.

- (1) The same harmonics that are present in the current wave are also present in the voltage wave.
- (2) The current harmonics lead the corresponding voltage harmonics by 90° , *measured on the scale of these harmonics*.
- (3) The amplitudes of corresponding current and voltage harmonics do not bear a constant ratio. But the ratio increases as the order of harmonic increases; or:

$$(I_{m3}/E_{m3}) > (I_{m1}/E_{m1}) \quad (24)$$

If an ammeter and a voltmeter were connected in the circuit, it would be found that the ratio of the effective values of the non-sine Emf. and current is different from the ratio of the effective values of any harmonic voltage and current; or:

$$X_c = (E/I) \neq E_n/I_n \quad (25)$$

These physical facts may be interpreted mathematically as follows.

(b) *Mathematical Theory.* — The instantaneous charge q , acquired by a capacitance C , is:

$$q = Ce \quad (26)$$

Remembering that $i = dq/dt$ and substituting in Eq. (26) the value of e as given by Eq. (5b) we have:

$$\begin{aligned} i = (dq/dt) &= (C de/dt) = \sum (Cn\omega_1)E_{mn} \cos n(\omega_1 t + \alpha_n) \quad (a) \\ &= \sum I_{mn} \sin n(\omega_1 t + \alpha_n + 90^\circ/n) \quad (b) \end{aligned} \quad (27)$$

where:

$$I_{mn} = Cn\omega_1 E_{mn} = n(2\pi f_1 C)E_{mn} = nB_{c1}E_{mn} = B_{cn}E_{mn} \quad (28)$$

Dividing both sides of Eq. (28) by $\sqrt{2}$ we have:

$$\begin{aligned} I_n &= B_{cn}E_n = nB_{c1}E_n \quad (a) \\ E_n &= X_{cn}I_n = (X_{c1}/n)I_n \quad (b) \end{aligned} \quad (29)$$

where:

$$\begin{aligned} B_{cn} &= nB_{c1} = 2\pi n f_1 C \quad (a) \\ X_{cn} &= (X_{c1}/n) = 1/(2\pi n f_1 C) \quad (b) \end{aligned} \quad (30)$$

Comparing Eqs. (5b) and (27b) we conclude that the same harmonics which are present in the voltage wave are also present in the current wave and that the current harmonics lead the corresponding voltage harmonics by 90° measured to the scale of these harmonics.

Eq. (30b) shows that *the capacitive reactance of the circuit to a given harmonic is inversely proportional to the order of that harmonic.*

Hence, for the same Emf., the higher the order of the harmonic the greater would be the current. This fact is further confirmed by Eq. (28) which explains why the third harmonic current (Fig. 4c) is almost equal to the first harmonic current although the third harmonic voltage is much smaller than the first harmonic Emf.

As in the case of inductive reactance, capacitive reactance to non-sine waves has no physical existence. Each harmonic has its own reactance. We may, however, define the equivalent capacitive reactance of a circuit to a non-sine wave as:

$$X_c = E/I = \left[\sum E_n^2 / \sum I_n^2 \right]^{1/2} = \left[\sum (X_n I_n)^2 / \sum I_n^2 \right]^{1/2} \quad (31)^*$$

5. A Series Circuit Consisting of R and $\mathcal{L}$ and a Non-Sine Emf. e . — Consider a series circuit which comprises a resistance and an inductance. Let the instantaneous value of the non-sine current flowing through it be:

$$i = \sum I_{mn} \sin n(\omega_1 t + \beta_n) \quad (32)$$

Let it be required to determine the Emf. which causes such a steady flow of current.

By Kirchhoff's Emf. Law we have:

$$e = Ri + \mathcal{L}(di/dt) \quad (33)$$

Substituting Eq. (32) in (33) and differentiating we obtain:

$$\left. \begin{aligned} e &= \sum RI_{mn} \sin n(\omega_1 t + \beta_n) + \sum n\omega_1 \mathcal{L} I_m \cos n(\omega_1 t + \alpha_n) & (a) \\ &= \sum RI_{mn} \sin n(\omega_1 t + \beta_n) + \sum X_{Ln} I_m \sin n(\omega_1 t + \alpha_n + 90^\circ/n) & (b) \end{aligned} \right\} \quad (34)$$

Adding the two components of the n th harmonic in Eq. (34b) and then taking the summation for all harmonics we obtain:

$$\left. \begin{aligned} e &= \sum Z_n I_{mn} \sin n(\omega_1 t + \beta_n + \phi_n/n) & (a) \\ &= \sum E_{mn} \sin n(\omega_1 t + \beta_n + \phi_n/n) & (b) \end{aligned} \right\} \quad (35)$$

where:

$$\left. \begin{aligned} Z_n &= \sqrt{R^2 + X_{Ln}^2} & (a) \\ E_{mn} &= Z_n I_{mn} & (b) \\ \tan \phi_n &= X_{Ln}/R & (c) \end{aligned} \right\} \quad (36)$$

Hence a non-sine current of the form given by Eq. (32) is caused by a non-sine voltage consisting of the same harmonics as those comprised in the current wave. The amplitude of the n th voltage harmonic

* See note following Eq. (23).

is expressed by Eq. (36*b*) and the phase angle between a current harmonic and the corresponding voltage harmonic is expressed by Eq. (36*c*).

Conversely a non-sine Emf. impressed on a series circuit consisting of R and $\mathcal{L}$ produces a steady current having the same number of harmonics as those comprised in the Emf. wave. Moreover, each of these harmonics has an amplitude expressed by Eq. (36*b*) and lags the corresponding harmonic of the impressed voltage by an angle ϕ_n . This phase angle is measured on the scale of the n th harmonic and has a value given by Eq. (36*c*).

Comparing Eqs. (35) and (36) with Eqs. (6), (8) and (9) in Chapter VII we conclude that the treatment of circuits subjected to non-sine waves is not essentially different from that for circuits acted upon by sine Emfs. In fact each harmonic of the non-sine wave acts as though it were alone and obeys the laws deduced for sine waves.

A numerical problem (No. 5*a*) is solved in an effort to render this treatment clear. (See problems at the end of this chapter.)

6. A Series Circuit Consisting of Resistance R , Inductance $\mathcal{L}$, and Capacitance C , and a Non-Sine Emf. e . — Let the non-sine current flowing through a circuit, consisting of R , $\mathcal{L}$, and C , in series, be expressed by Eq. (32) and let it be required to determine the Emf. which causes this current.

By Kirchhoff's Emf. Law the instantaneous value of the impressed voltage is equal to the sum of the Emf. drops across the various characteristics; or:

$$e = Ri + \mathcal{L}(di/dt) + \int i dt/C \quad (37)$$

Substituting Eq. (32) in (37) we have:

$$\left. \begin{aligned} e &= \sum RI_{mn} \sin n(\omega_1 t + \beta_n) + \sum n\omega_1 \mathcal{L} I_{mn} \cos n(\omega_1 t + \beta_n) \\ &\quad - \sum (I_{mn}/n\omega_1 C) \cos n(\omega_1 t + \beta_n) + E_0 \end{aligned} \right\} (a) \quad (38)$$

or $e = \sum RI_{mn} \sin n(\omega_1 t + \beta_n)$

$$+ \sum (X_{Ln} - X_{cn}) I_{mn} \sin n(\omega_1 t + \beta_n + 90^\circ/n) + E_0 \quad (b)$$

where E_0 is a constant of integration and physically represents a continuous Emf. of any value whatever, in series with the non-sine Emf. This means that it is possible in such a circuit to obtain a non-sine current, consisting of only harmonics without a D.C. component, from a non-sine Emf. which comprises these harmonics and a continuous Emf. That such is the case may be readily appreciated when it is remembered that, under established conditions, a continuous Emf. causes no flow of current because the capacitor acts as an open circuit.

Hence the only current which flows through the circuit is the non-sine current represented by Eq. (32).*

Combining the two series of sine waves in Eqs. (38) into one series we have:

$$\left. \begin{aligned} e &= E_0 + \sum Z_n I_{mn} \sin n(\omega_1 t + \beta_n + \phi_n/n) & (a) \\ &= E_0 + \sum E_{mn} \sin n(\omega_1 t + \beta_n + \phi_n/n) & (b) \end{aligned} \right\} \quad (39)$$

where:

$$\left. \begin{aligned} E_{mn} &= I_{mn} \sqrt{R^2 + (X_{Ln} - X_{Cn})^2} = I_{mn} Z_n & (a) \\ \tan \phi_n &= (X_{Ln} - X_{Cn})/R & (b) \end{aligned} \right\} \quad (40)$$

Eq. (39) states that a non-sine current, of instantaneous value given by Eq. (32), which flows through a series circuit of R , $\mathcal{L}$, and C , is produced by a non-sine Emf. having a constant component E_0 and the same number of harmonics as are contained in the current wave. Each harmonic has an amplitude E_{mn} expressed by Eq. (40a) and is out of phase with the corresponding current harmonics by an angle ϕ_n (Eq. 40b) *measured to the scale of that harmonic*.

Conversely, if a periodic Emf. is impressed on a series circuit consisting of R , $\mathcal{L}$, and C , then a periodic current is caused to flow through that circuit. This current has all the harmonics contained in the Emf. wave but does not possess a continuous current component. The amplitude of any current harmonic is expressed by Eq. (40a) and its initial angle is given by Eq. (40b).

A question arises as to what is the impedance of the circuit. The answer is that such an impedance does not exist. Each voltage harmonic has its own impedance and sends its own current through the circuit as though it were acting alone. We may, however, define an equivalent impedance Z such that:

$$Z = (E/I) = \left[\sum E_n^2 / \sum I_n^2 \right]^{1/2} \quad (41)^\dagger$$

These facts are illustrated by the numerical problem No. 6a, in the list at the end of the chapter.

7. Voltage Resonance in a Series Circuit Subjected to a Non-Sine Emf. — Resonance in a series circuit is discussed in Art. 8, Chapter VII, and the resonant frequency is defined as:

$$f_r = 1/(2\pi\sqrt{\mathcal{L}C}) \quad (42)$$

* Some authors assert that the term E_0 (Eqs. 38) vanishes after a time. This concept is entirely wrong. If a continuous Emf. E_0 is present in the circuit, there is nothing that can make it vanish except its actual removal by physical means. Time has nothing whatever to do with its vanishing.

† See note following Eq. (23).

Now when a non-sine wave is impressed on a series circuit it is possible that one of its harmonics may possess just that resonant frequency. Under these conditions the current due to that harmonic is limited solely by the resistance of the circuit. If R is small, the current due to that particular harmonic and hence the drops across $\mathcal{L}$ and C may reach exorbitant values. It is advisable, therefore, to make preliminary computations and determine beforehand whether the characteristics of the circuit are such as to resonate with any of the harmonics contained in the impressed voltage.

8. Parallel Circuits Consisting of Conductance G , Inductance $\mathcal{L}$, Capacitance C and a non-sine Emf. e . — Let a non-sine voltage, whose instantaneous value e is:

$$e = \sum E_{mn} \sin n(\omega_1 t + \alpha_n) \quad (43)$$

be impressed on a parallel circuit consisting of a conductance G , inductance $\mathcal{L}$, and capacitance C . The instantaneous value of the currents flowing in each of the three branches are, see Eqs. (12), (19) and (27):

$$\left. \begin{aligned} i_g &= \sum G E_{mn} \sin n(\omega_1 t + \alpha_n) & (a) \\ i_L &= \sum (B_L E_{mn}) \sin n(\omega_1 t + \alpha_n - 90^\circ/n) & (b) \\ i_c &= \sum B_c E_{mn} \sin n(\omega_1 t + \alpha_n + 90^\circ/n) & (c) \end{aligned} \right\} \quad (44)$$

The non-sine current is by Kirchhoff's current law:

$$\left. \begin{aligned} i &= i_g + i_L + i_c & (a) \\ \text{or } i &= \sum G E_{mn} \sin n(\omega_1 t + \alpha_n) & (b) \\ &+ \sum (B_{Ln} - B_{cn}) E_{mn} \sin n(\omega_1 t + \alpha_n - 90^\circ/n) & (b) \end{aligned} \right\} \quad (45)$$

Remembering that the sum of two sine waves is a sine wave and adding the two sine waves of the n th harmonic, in Eq. (45b), then forming the summation for all harmonics we have:

$$i = \sum I_{mn} \sin n(\omega_1 t + \alpha_n - \phi_n/n) \quad (46)$$

where:

$$\left. \begin{aligned} I_{mn} &= E_{mn} \sqrt{G^2 + (B_{Ln} - B_{cn})^2} & (a) \\ \tan(\phi_n) &= (B_{Ln} - B_{cn})/G & (b) \end{aligned} \right\} \quad (47)$$

The equivalent admittance of the circuit is:

$$Y = I/E = \left[\sum I_n^2 / \sum E_n^2 \right]^{1/2} \quad (48)$$

The admittance Y in Eq. (48) is a fictitious quantity which has no physical existence and which cannot be computed from the known characteristics R , $\mathcal{L}$, and C .

This definition of equivalent admittance arises from the tendency (in practice) to replace the non-sine waves of Emf. and current by equivalent sine waves (that is, sine waves whose effective values are equal to the effective values of the non-sine waves). It must be understood that the value of the equivalent admittance given by Eq. (48) remains constant if:

- (a) The same harmonics exist in the Emf. wave.
- (b) The ratios of the amplitudes of these harmonics are the same.
- (c) The ratio ($\mathcal{E}/C$) remains constant.

The use of equivalent sine waves and equivalent characteristics should (in the author's opinion) be discouraged. The limitations imposed, the deviation from physical facts, and the erroneous (sometimes absurd) results that are obtained furnish enough arguments against this practice.

9. Current Resonance in a Parallel Circuit Subjected to a Non-Sine Emf. — Resonance in a parallel circuit is discussed in Art. 8, Chapter VIII, and the resonant frequency is defined by Eq. (42) above. It is possible that one of the harmonics, of the non-sine wave impressed on a parallel circuit, may have a frequency which is equal to the resonant frequency. Under these conditions that harmonic exhibits all the symptoms of current resonance.

10. Series-Parallel Circuits. — The problem of finding the current and voltage relations in a circuit like that represented by Fig. 2, Chapter X, lends itself to solution if the equivalent series or parallel circuit for each harmonic frequency is first determined. Such is the case (1) because the circuit characteristics (X_L and X_C) are different for different frequencies and (2) because each harmonic behaves as though it were acting alone.

The current, that would flow in a series-parallel circuit due to an impressed non-sine voltage wave can be determined in steps as follows:

(1) Convert the series-parallel circuit to an equivalent series or parallel circuit for each of the harmonics appearing in the non-sine wave. This is done exactly as is demonstrated for sine waves (Chapter X). Different values of X_{Ln} , B_{Ln} , X_{Cn} and B_{Cn} must be used for the various harmonics.

(2) Determine the instantaneous value of the current that would flow in that equivalent series or parallel circuit by the use of Eqs. (35) and (36) or Eqs. (46) and (47).

(3) The effective values of the current and voltage can be computed by using Eq. (6).

(4) The equivalent impedance or admittance for the specified non-sine wave can then be obtained by using Eq. (41) or (48).

11. Emf.-Impedance Combinations. — Consider a circuit as that shown in Fig. 7, Chapter X. Let the Emf. sources be non-sinusoidal. The problem of finding the currents in each branch consists of first determining the current for each harmonic as was demonstrated in Chapter X. Then the instantaneous current in each branch is simply the sum of the instantaneous currents due to the various harmonics in that branch.

12. Power in Circuits Subjected to Non-Sine Waves. — (a) *Instantaneous and* (b) *Average Power.* — Consider a non-sine voltage whose instantaneous value e is expressed by Eq. (43) and let the current due to this voltage be expressed by Eq. (46).

(a) *The Instantaneous Power is:*

$$p = ei = \left[\sum E_{mn} \sin n(\omega_1 t + \alpha_n) \right] \left[\sum I_{mn} \sin n(\omega_1 t + \alpha_n \pm \phi_n/n) \right] \quad (49)$$

(b) *The Average Power is:*

$$P = (1/T) \int_0^T p dt = (1/T) \int_0^T \sum [E_{mn} I_{mn} \sin n(\omega_1 t + \alpha_n) \sin n(\omega_1 t + \alpha_n \pm \phi_n/n)] dt \\ + (1/T) \int_0^T \sum [E_{mn} I_{mu} \sin n(\omega_1 t + \alpha_n) \sin u(\omega_1 t + \alpha_u \pm \phi_u/u)] dt \quad (50)$$

where $u \neq n$.

The second integral in Eq. (50) is identically equal to zero (see Theorem II, Appendix I). Hence the average power is represented by the first integral; or:

$$P = \sum E_n I_n \cos \phi_n = \sum I_n^2 R \quad (51)$$

A significant fact about Eq. (51) is that each harmonic voltage produces power with its own harmonic current and with no other current. The total average power is just the algebraic sum of the powers due to the separate harmonics.

(c) *Apparent Power.* — Apparent power is defined in Chapter XII as the product of the effective value of the Emf. and current or:

$$P_a = EI = \sqrt{\sum E_n^2} \sqrt{\sum I_n^2} \quad (52)$$

(d) *Power Factor.* — As has been demonstrated above, each harmonic has its own power factor. Strictly speaking, therefore, there exists no such thing as the power factor of a circuit with a non-sine Emf. and current. Power factor is arbitrarily defined as the ratio of the average power to the apparent power or:

$$\text{P.f.} = (P/P_a) = \left[\sum E_n I_n \cos \phi_n \right] / \sqrt{\sum E_n^2} \sqrt{\sum I_n^2} \quad (53)$$

According to this definition the power factor for non-sine waves may be less than unity even with a pure resistance circuit. Thus, for a pure resistance circuit $\cos \phi_n = 1$. Hence Eq. (53) becomes:

$$\text{P.f.} = \left[\left(\sum E_n I_n \right) / \sqrt{\sum E_n^2 \sum I_n^2} \right] \leq 1 \quad (54)$$

Eq. (54) is equal to unity if and only if the resistance of the circuit does not vary with the harmonic; that is, if the skin effect is neglected. A proof of this statement may be found in Theorem IV, Appendix I. If, however, the skin effect is not neglected the P.f. for a pure resistance circuit becomes less than unity when that circuit is subjected to a non-sine Emf. This might be a startling result to those who are accustomed to regard the power factor as a measure of the relative amount of power supplied and power dissipated as Joulean heat. In circuits involving sine Emfs. and currents this physical concept of power factor is valid. With non-sine waves, the definition of power factor is purely a matter of convention and has no physical basis whatever.

(e) *Reactive Power (or Volt-Amperes)*. — The generally accepted definition of reactive power is:

$$P_x = \sqrt{P_a^2 - P^2} \quad (55)$$

There is no physical basis for so defining reactive power. The justification for this definition lies in the fact that both P_a and P are measurable quantities. Therefore, P_x can be easily computed from Eq. (55).

The problem of establishing sound definitions, of reactive power and power factor, which are based on physical facts is important both from a scientific and a financial view point. No scientist can be satisfied with an arbitrary definition of power factor or of reactive power as given by Eqs. (53) and (55). There must be a physical basis underlying any definition of any physical phenomenon. Financially speaking there is the problem of how much to charge a customer for power consumed and for power borrowed and returned (reactive power). Before a rate can be established, the fundamental question, of how much reactive power has been borrowed or returned, has to be settled. Such a settlement does not consist in adopting an arbitrary definition but rather in studying the underlying physical facts. It is with this two-fold object in view that the Rumanian questionnaire on reactive power referred to on page 160 has been prepared and distributed. The A.I.E.E. has appointed a special committee to study this problem. The reports of this committee and of the International Commission will be watched with interest.

13. Energy in Circuits Subjected to Non-Sine Emfs. — The general expression for the energy supplied during an interval $t = t_2 - t_1$ is:

$$W = \int_{t_1}^{t_2} p \, dt = \int_{t_1}^{t_2} ei \, dt \quad (56)$$

Applying this definition to non-sine waves we have:

$$W = \int_{t_1}^{t_2} \left[\sum E_{mn} \sin n(\omega_1 t + \alpha_n) \sum I_{mn} \sin n(\omega_1 t + \alpha_n \pm \phi_n/n) \right] dt \quad (57)$$

PROBLEMS

1. A non-sine wave consists of a fundamental, a 10 per cent* third harmonic, and a 7 per cent fifth harmonic. Moreover, at the instant $t = 0$, the fundamental, third, and fifth harmonics have instantaneous values $e_1 = 0.5 E_{m1}$, $e_3 = -0.3 E_{m3}$, and $e_5 = -0.4 E_{m5}$; and the slopes of the waves are all positive.

- (a) Find the phase angles α_1 , α_3 , and α_5 .
- (b) Express the instantaneous value of the non-sine wave in the form of a Fourier's series.
- (c) Draw the harmonics as well as the non-sine wave for half a cycle, showing the instant $t = 0$ and the phase angles α_1 , α_3 , and α_5 .
- (d) Determine the effective value of the non-sine wave.

2a. The fundamental voltage, in problem 1, has an amplitude $E_{m1} = 15$ volts and a frequency of 5000 cycles per second. Give the instantaneous value of the current that would flow when this voltage is impressed on a circuit having a non-inductive resistance of 5 ohms.

2b. A circuit consisting of a resistance R ohms has an effective current of 5 amp. flowing through it. Upon analysis it was found that the Emf. wave can be expressed by the following series:

$$e = 10 \sin (\omega_1 t + 15^\circ) + 5 \sin 3 (\omega_1 t + 5^\circ)$$

- (a) Determine the resistance of the circuit.
- (b) Express the instantaneous value of the current by a Fourier's series.
- (c) Give the effective values of the Emf. and current.

2c. Analysis of a current oscillogram for a resistance circuit of 10 ohms gives the following data:

$I_{m1} = 10 \text{ amp.}$	$I_{m3} = 2 \text{ amp.}$	$I_{m5} = 1 \text{ amp.}$
$\beta_1 = 5^\circ$	$\beta_3 = 10^\circ$	$\beta_5 = 15^\circ$

- (a) Write the Fourier's series for the instantaneous value of the current.
- (b) Express the instantaneous value of the impressed Emf. by a Fourier's series.
- (c) Determine the effective values of the Emf. and current.

* The amplitudes of the harmonics are frequently expressed in terms of the amplitude of the fundamental. Thus when it is stated that a wave has a 5 per cent ninth harmonic it means that $E_{m9} = 0.05 E_{m1}$.

3a. A circuit has an inductance of 10 mh. What is its inductive reactance to the fundamental, 2nd, 3rd, 4th and 5th harmonics if the frequency of the fundamental is 1000 cycles per second.

3b. A circuit of inductance $\mathcal{L} = 0.5$ mh. is subjected to the non-sine Emf. given in problem 2b.

- (a) Give the instantaneous value of the current flowing through the circuit, assuming that the frequency of the fundamental is 5000 cycles per second.
- (b) Give the effective value of the current.

3c. The Emf. specified in problem 2b causes a current

$$i = 2 \sin (\omega_1 t - 75^\circ) + I_{m3} \sin 3 (\omega_1 t - \beta_3).$$

Assuming that the frequency of the fundamental Emf. wave is 10,000 cycles per second,

- (a) Determine the inductance of the circuit.
- (b) Compute I_{m3} .
- (c) Compute β_3 .
- (d) Determine the effective value of the current.

3d. Analysis of a current oscillogram taken for a circuit having an inductance $\mathcal{L} = 15$ mh. gives the following data:

$$\begin{array}{ll} I_{m1} = 6 \text{ amp.} & I_{m5} = 2 \text{ amp.} \\ \beta_1 = 15^\circ & \beta_5 = 5^\circ \end{array}$$

- (a) Give the Fourier's series for the instantaneous value of the current.
- (b) Assuming that the fundamental frequency is 1000 cycles per second, express the impressed voltage by a Fourier's series.
- (c) Determine the effective values of the Emf. and current.

4a. A circuit has a capacitance of 3 μ f. What is its capacitive reactance to the fundamental, 2nd, 3rd, 4th and 5th harmonics if the frequency of the fundamental is 50,000 cycles per second.

4b. A circuit having a capacitance $C = 8$ μ f. is subjected to the non-sine Emf. given in problem 2b.

- (a) Give the instantaneous value of the current flowing through the circuit, assuming that the frequency of the fundamental is 20,000 cycles per second.
- (b) What is the effective value of the current?

4c. The Emf. specified in problem 2b causes a current

$$i = 0.5 \sin (\omega_1 t + 75^\circ) + I_{m3} \sin 3 (\omega_1 t + \beta_3)$$

Assuming that the frequency of the fundamental Emf. wave is 100,000 cycles per second,

- (a) Determine the capacitance of the circuit.
- (b) Compute I_{m3} .
- (c) Compute β_3 .
- (d) Determine the effective value of the current.

4d. A current oscillogram taken for a circuit having capacitance $C = 4$ μ f. was analyzed giving the following results:

$$\begin{array}{ll} I_{m1} = 0.2 \text{ amp.} & I_{m2} = 0.3 \text{ amp.} \\ \beta_1 = 5^\circ & \beta_2 = 10^\circ \end{array}$$

- (a) Express the instantaneous value of the current by a Fourier's series.
- (b) Assuming that the fundamental frequency is 80,000 cycles per second, give the Fourier's series of the impressed voltage.
- (c) Compute the effective value of the Emf. and current.

5a. An Emf.:

$$e = 150 \sin (120 \pi t + 25^\circ) + 25 \sin 3 (120 \pi t + 10^\circ) + 15 \sin 5 (120 \pi t + 4^\circ)$$

is applied to a series circuit having a resistance $R = 5$ ohms and an inductance $\mathcal{L} = 15$ mh.

- (a) Give the Fourier's series for the steady current flowing through the circuit.
- (b) Determine the effective values of the Emf. and current.

Solution. —

$$\begin{array}{lll} X_1 = 2 \pi f_1 \mathcal{L} = 5.66 & X_3 = 3 x_1 = 17.0 & X_5 = 5 x_1 = 28.3 \\ Z_1 = \sqrt{R^2 + X_1^2} = 7.56 & Z_3 = 17.75 & Z_5 = 28.5 \\ \phi_1 = \tan^{-1} (X_1/R) = 48^\circ 40' & \phi_3 = 73^\circ 40' & \phi_5 = 80^\circ 0' \\ & \phi_3/3 = 24^\circ 20' & \phi_5/5 = 16^\circ \\ I_{m1} = (150/7.56) = 19.85 & I_{m3} = (25/17.75) = 1.41 & I_{m5} = (15/28.5) = 0.526 \end{array}$$

$$i = 19.85 \sin (\omega t + 25 - 48^\circ 40') + 1.41 \sin 3 (\omega t + 10^\circ - 24^\circ 20') + 0.526 \sin 5 (\omega t + 4^\circ - 16^\circ)$$

$$E = (\sqrt{150^2 + 25^2 + 15^2})/2 = 108.25$$

$$I = (\sqrt{(19.85)^2 + (1.41)^2 + (0.526)^2})/2 = 14.11$$

5b. A non-sine voltage wave of instantaneous value e produces a current of instantaneous value i such that:

$$\begin{aligned} e &= 15 \sin (1200 \pi t + 25^\circ) + 2 \sin 2 (1200 \pi t + 5^\circ). \\ i &= 10 \sin (1200 \pi t + \beta) + I_{m2} \sin 2 (1200 \pi t - 10^\circ). \end{aligned}$$

- (a) Compute the resistance and inductance of the circuit.
- (b) Find the angle β .
- (c) Determine I_{m2} .
- (d) Give the effective values E and I .
- (e) Compute the equivalent impedance of the circuit.

6a. A series circuit consists of a resistance $R = 80$ ohms, inductance $\mathcal{L} = 15$ mh. and capacitance $C = 15$ μ f. The oscillogram of the impressed Emf. wave may be expressed by the following Fourier's series.

$$e = 20 + 110 \sin (120 \pi t + 25^\circ) + 15 \sin 3 (120 \pi t + 20^\circ).$$

It is required to determine the steady current which flows through the circuit.

Solution. — Under steady conditions, the constant term of 20 volts produces no current. Its presence simply results in the accumulation of a charge on the plates of the condenser. The Emf. across the condenser, due to this charge, is equal and opposite to the constant term (or 20 volts). We shall therefore concern ourselves only with the two harmonics. We have:

$$\begin{aligned} X_{L1} &= 5.66 \text{ ohms} & X_{L3} &= 3 x_{L1} = 17 \text{ ohms} \\ X_{C1} &= 176.5 \text{ ohms} & X_{C3} &= (1/3) x_{C1} = 58.8 \text{ ohms.} \\ Z_1 &= \sqrt{80^2 + (5.66 - 176.5)^2} = 188.7 \text{ ohms} \end{aligned}$$

$$Z_3 = \sqrt{80^2 + (17 - 58.8)^2} = 90.3 \text{ ohms}$$

$$\phi_1 = -64^\circ 50' \quad \phi_3 = -27^\circ 40' \quad (\phi_3/3) = -9^\circ 13'$$

$$I_{m1} = (110/188.7) = 0.583 \quad I_{m3} = (15/90.3) = 0.166$$

$$(\beta_1 + \phi_1) = 25^\circ \quad \text{Therefore} \quad \beta_1 = 25^\circ - \phi_1 = 89^\circ 50'$$

$$(\beta_3 + \phi_3/3) = 20^\circ \quad \text{Therefore} \quad \beta_3 = 29^\circ 13'$$

$$i = 0.583 \sin (\omega_1 t + 89^\circ 50') + 0.166 \sin 3 (\omega_1 t + 29^\circ 13')$$

6b. Develop the current-voltage relations for a series circuit consisting of R , C and a non-sine Emf. e ; and show that the results so obtained check with Eqs. (38), (39), and (40), provided that X_{Ln} in these equations is made equal to zero.

6c. A series circuit consists of a resistance $R = 4$ ohms and a capacitance $C = 5 \mu\text{f}$. The Emf. impressed on the circuit is of the form:

$$e = 10 \sin (\omega_1 t + 18^\circ) + 4 \sin 3 (\omega_1 t + 10^\circ)$$

The frequency of the fundamental is 10,000 cycles per second.

- (a) Compute the impedance of the circuit to the fundamental and the third harmonic.
- (b) Determine the phase angles ϕ_1 and ϕ_3 .
- (c) Find the amplitudes of the fundamental and third harmonic currents.
- (d) Write the Fourier's series for the current flowing through the circuit.
- (e) Determine the effective values of the Emf. and current.
- (f) Compute the equivalent impedance of the circuit.

7. At what frequency will the circuit specified in problem 6a exhibit resonance?

8. A parallel circuit consists of a conductance $G = 0.5$ mho, a condenser $C = 3 \mu\text{f}$, and an inductance $\mathcal{L} = 0.5$ mh. This circuit is subjected to an Emf.:

$$e = 10 \sin (\omega_1 t + 15^\circ) + 2 \sin 4 (\omega_1 t + 25^\circ)$$

The frequency of the fundamental is 8000 cycles per second.

- (a) Give the instantaneous values of the currents flowing through the conductance, inductance, and capacitance.
- (b) Determine the instantaneous value of the total current flowing through the circuit.
- (c) Give the effective value of the impressed Emf. and of the total circuit current.
- (d) Compute the equivalent admittance of the circuit.

9. What is the resonant frequency of the circuit specified in problem 8?

10. Let the characteristics of the series parallel circuit, Fig. 8, Chapter X, be true for fundamental frequency; and let the Emf. impressed on the circuit be:

$$e = 150 \sin (120 \pi t + 30^\circ) + 5 \sin 3 (120 \pi t + 20^\circ)$$

- (a) Compute the circuit current due to the fundamental Emf.
- (b) Determine the circuit current due to the third harmonic Emf.
- (c) Give the instantaneous value of the non-sine current flowing through the circuit.
- (d) Compute the effective values of the Emf. and current
- (e) Determine the equivalent impedance of the circuit.

11. Referring to Fig. 7, Chapter X, let the instantaneous voltages be:

$$e_{12} = 150 \sin (\omega_1 t + \alpha_1) + 30 \sin 3 (\omega_1 t + \beta_3)$$

$$e_{13} = 75 \sin (\omega_1 t + \beta_1) + 50 \sin 2 (\omega_1 t + \beta_2)$$

$$e_{34} = 50 \sin (\omega_1 t + \theta_1) + 20 \sin 5 (\omega_1 t + \theta_5)$$

where $\omega_1 = 120 \pi$

Let the impedances at fundamental frequency be:

$$Z_a = 5 + j3$$

$$Z_d = 7 + j8$$

$$Z_b = 2 + j4$$

$$Z_e = 9 - j1$$

$$Z_c = 6 - j7$$

$$Z_f = 2 + j3$$

(a) Determine the current flowing through each branch of the circuit due to each harmonic.

(b) Find the effective value of the non-sine current flowing in each branch.

12a. Referring to any of the circuit problems given above, compute:

(a) the instantaneous power.

(d) the equivalent P.f.

(b) the average power.

(e) the reactive power.

(c) the apparent power.

12b. Make a study of the physical phenomena (see Chapter XII) underlying our concept of power and endeavor to obtain definitions of reactive power and of P.f. which are based on physical facts.

13. Compute the energy supplied to the circuit specified in problem 5a during an interval of 0.01 second from the instant $t = 0$.

CHAPTER XV

POLYPHASE CIRCUITS

PART A — BALANCED CIRCUITS

1. Polyphase Alternators. — The reader is referred to Art. 2, Chapter VI, where a description of a single-phase alternator is given. A two-phase alternator (Fig. 1a) differs from a single-phase machine in that it comprises two independent windings instead of just one. Thus let the armature (Fig. 1a) have eight equidistant slots. Let these slots be divided into two groups by marking them alternately 1 and 2. Let insulated conductors be placed in these slots and so connected that all conductors occupying slots 1 are connected in series and all those occupying slots 2 are also connected in series, so as to form two independent windings with terminals 11' and 22' respectively. If the flux distribution along the air-gap is sinusoidal, then the voltage induced in any one conductor will also be sinusoidal. Furthermore the sum of the Emfs. induced in the first group of conductors is equal to the terminal voltage $E_{11'}$ and the sum of the Emfs. induced in the second group of conductors is equal to the terminal voltage $E_{22'}$.

An inspection of Fig. 1a shows that at the instant when the first group of conductors are under the centers of the poles, the second group of conductors fall in the neutral spaces between two adjacent poles. In other words, when the flux cutting the first group of conductors is a maximum the flux cutting the second group is zero; and vice versa. Now the induced voltage in a conductor is proportional to the density of the flux which cuts that conductor ($e = Bvd$). Hence when the voltage induced in the first group of conductors is a maximum, that induced in the second group is zero; and vice versa. There exists, therefore, a phase displacement in time of 90° (Fig. 2a) between the induced voltages $e_{11'}$ and $e_{22'}$.

The first group of conductors (Fig. 1a) forms a winding which is called *phase 1*, and the second group forms another winding which is known as *phase 2*. The machine is a *two-phase* alternator.

Next let each pole pitch (covering π electrical radians, Fig. 1b) be divided into three equal parts such that the points of division are 60° electrical degrees apart. Let slots be cut at these points of division and let insulated conductors be placed in these slots. Fig. 1b thus has twelve equidistant slots. Divide these slots into three groups by

marking them consecutively 1, 3, 2 (not 1, 2, 3). Connect in series all the conductors occupying slots 1 and obtain a winding with the terminals 1'1. Do the same with the conductors occupying slots 2 and slots 3 respectively and obtain two windings with terminals 2'2 and 3'3'.

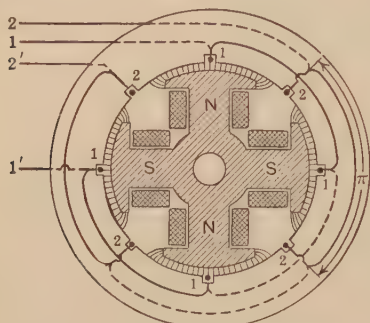


Fig. 1a. Two-phase alternator.

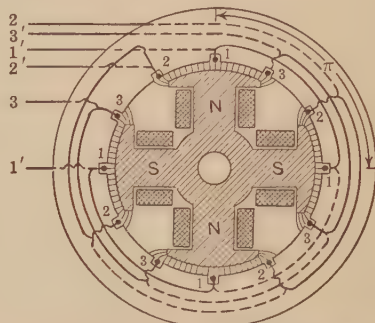


Fig. 1b. Three-phase alternator.

The alternator is then said to be a three-phase alternator having three separate windings with six terminals. The induced voltages $e_{1'1}$, $e_{2'2}$ and $e_{3'3'}$, measured at these terminals, are displaced from each other (in time) by 60° , as is shown in Fig. 2b.

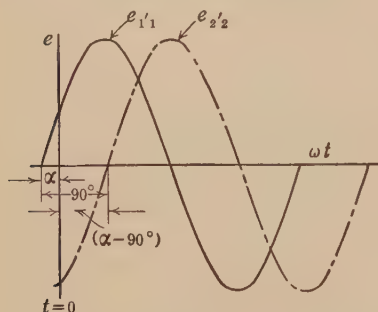


Fig. 2a

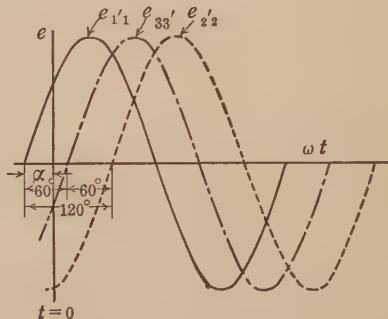


Fig. 2b

Fig. 2a. Emf. waves produced by the two-phase alternator Fig. 1a.

Fig. 2b. Emf. waves produced by the three-phase alternator Fig. 1b.

Although alternators are generally constructed three-phase the reader can see how an electric machine may be wound for any number (p) of phases by having an independent winding for each phase. The conductors constituting these separate windings are generally so displaced in space that there exist $2\pi/p$ electrical radians between any two corresponding conductors of two consecutive phases. Rotary converters furnish a practical illustration of electrical machines built for six or more phases.

2. Emf. Waves and Vector Diagrams for Polyphase Sources. — The Emf. of each phase of a polyphase electrical machine shall be represented by a sine wave inserted within a circle. Thus Figs. 3*a*, 4*a*, and 5*a* represent respectively the Emfs. of a two-phase, a three-phase, and a four-phase machine. The magnitudes and phase relations of these Emfs. are represented by vector diagrams and oscillograms, as shown in Figs. 3*b*, 4*b*, and 5*b* and Figs. 3*c*, 4*c*, and 5*c* respectively.

Referring to Fig. 3*c* the instantaneous value expressions for the Emfs. induced in phases 1 and 2 respectively are:

$$\left. \begin{aligned} e_{1'1} &= E_m \sin (\omega t + \alpha) & (a) \\ e_{2'2} &= E_m \sin (\omega t + \alpha - 90^\circ) & (b) \end{aligned} \right\} \quad (1)$$

The vectors representing these Emfs. are:

$$\left. \begin{aligned} \mathbf{E}_{1'1} &= E \epsilon^{j\alpha} & (a) \\ \mathbf{E}_{2'2} &= E \epsilon^{j(\alpha - 0.5\pi)} & (b) \end{aligned} \right\} \quad (2)$$

The student can similarly write the instantaneous value expressions and the vector equations for Figs. 4 and 5.

Henceforth, we shall use only the vectors representing the Emf. waves and assume that the student can readily visualize the sine waves represented by these vectors.

3. Balanced Polyphase Emf. Sources. — A polyphase source (having p phases) is said to be balanced when:

- (a) The Emfs. of all the phases have equal effective values.
- (b) The vectors representing these Emfs. are equally displaced from each other so that the phase angle between any two consecutive vectors is $(360/p)$ degrees.

Figs. 3 and 4 represent unbalanced polyphase sources because the angle between the vectors representing the phase voltages are not equal to $360^\circ/p$. Thus in Fig. 3 the phase angle is 90° instead of $360/2 = 180^\circ$; and in Fig. 4 the phase angle is 60° instead of $360/3 = 120^\circ$.

Fig. 5 represents a balanced four-phase source. The phase voltages are equal in magnitude and are displaced from each other by $(360/4) = 90^\circ$.

The three-phase source, Fig. 4, could be balanced by reversing the terminals of the middle phase 33'. This means that the vector $\mathbf{E}_{33'}$ is changed to $\mathbf{E}_{3'3} = -\mathbf{E}_{33'}$. Moreover the sine wave $e_{33'}$ is changed to $e_{3'3}$. Under these conditions the effective values of the phase voltages are equal and the vectors representing the three phases ($\mathbf{E}_{1'1}$, $\mathbf{E}_{2'2}$, and $\mathbf{E}_{3'3}$) are displaced from each other by $(360/3) = 120^\circ$.

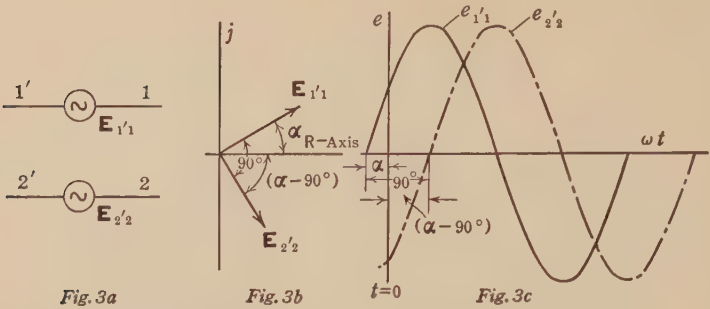


FIG. 3a. Emfs. of a two-phase source.
FIG. 3b. Vectors representing the Emfs., Fig. 3a.
FIG. 3c. Sine waves of the Emfs., Fig. 3a.

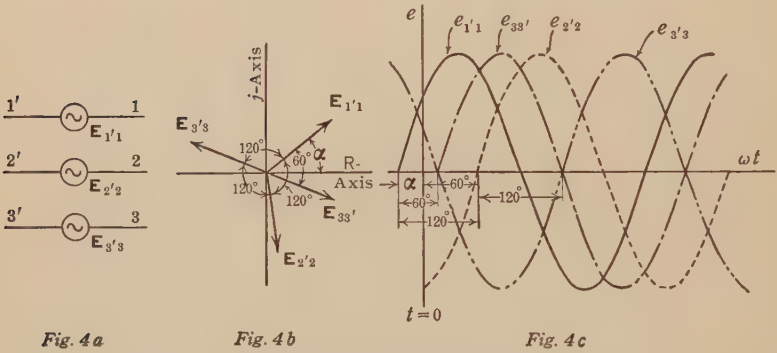


FIG. 4a. Emfs. of a three-phase source.
FIG. 4b. Vectors representing the Emfs., Fig. 4a.
FIG. 4c. Sine waves of the Emfs., Fig. 4a.

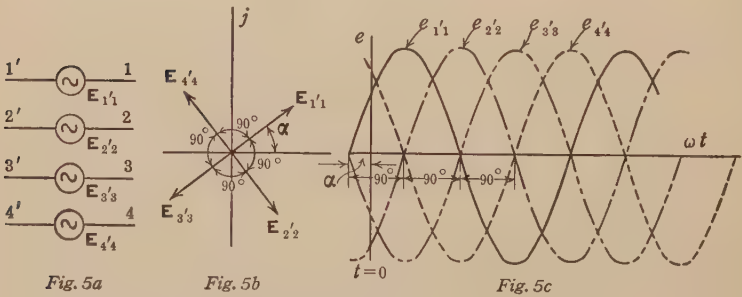


FIG. 5a. Emfs. of a four-phase source.
FIG. 5b. Vectors representing the Emfs., FIG. 5a.
FIG. 5c. Sine waves of the Emfs., Fig. 5a.

The instantaneous values of the Emfs. produced in the three phases (Fig. 4c) are:

$$\left. \begin{aligned} e_{1'1} &= E_m \sin (\omega t + \alpha) & (a) \\ e_{2'2} &= E_m \sin (\omega t + \alpha - 120^\circ) & (b) \\ e_{3'3} &= E_m \sin (\omega t + \alpha - 240^\circ) = E_m \sin (\omega t + \alpha + 120^\circ) & (c) \end{aligned} \right\} \quad (3)$$

The vectors representing these sine Emfs. are:

$$\left. \begin{aligned} \mathbf{E}_{1'1} &= E \epsilon^{j\alpha} & (a) \\ \mathbf{E}_{2'2} &= E \epsilon^{j(\alpha-2\pi/3)} & (b) \\ \mathbf{E}_{3'3} &= E \epsilon^{j(\alpha+2\pi/3)} & (c) \end{aligned} \right\} \quad (4)$$

This chapter is devoted to balanced circuits. We shall therefore reserve the treatment of the two-phase source to a later chapter and proceed with balanced sources.

4. Methods of Interconnecting Balanced Polyphase Sources. — The terminals of the separate phases of a polyphase Emf. source may either be left free as shown in Figs. 3a, 4a, and 5a, or they may be interconnected. There are two types of interconnection:

- (a) *The Star (or Y in the Case of Three-Phase).* — This type of connection is characterized by the presence of one point, 0 (Figs. 6a and 7a) which is common to all the interconnected phases.
- (b) *The Mesh (or Delta, Δ , in the Case of Three-Phase).* — This type of polyphase connection is characterized by the absence of a point which is common to all the phases. It consists of connecting unlike terminals (see Figs. 8a and 9a) of successive phases together thereby reducing the $2p$ terminals to p terminals.

We shall next develop the relations among the vector Emfs. in these two types of polyphase connections.

5. Emf. Relations in a Star-Connected Source. — (a) *The Three-Phase Star- or Y-Connection.* — Referring to Fig. 6a, let the terminals 1', 2' and 3' of a three-phase balanced source be connected together to a common point 0. There exist six different voltages: (1) three phase-voltages, and (2) three line-voltages.

(1) *The Three Phase-Voltages* are measured between the point 0 and the terminals 1, 2, and 3 respectively. Evidently these voltages are identical with the terminal Emfs. of the three phases. Thus:

$$\left. \begin{aligned} \mathbf{E}_{01} &= \mathbf{E}_{1'1} & (a) \\ \mathbf{E}_{02} &= \mathbf{E}_{2'2} & (b) \\ \mathbf{E}_{03} &= \mathbf{E}_{3'3} & (c) \end{aligned} \right\} \quad (5)$$

The terminal Emfs. of the alternator are equal and equally displaced from each other by 120° . But the phase-voltages are equal respectively to these Emfs. Hence the phase-voltages are equal and are displaced from each other by 120° (see Fig. 6b). Thus the sum of the phase-voltages is zero.

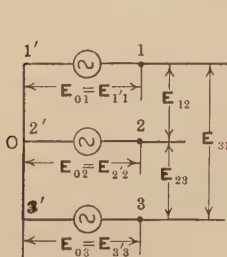


Fig. 6a

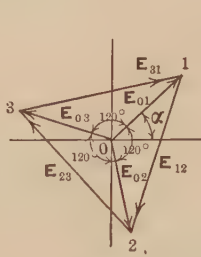


Fig. 6b

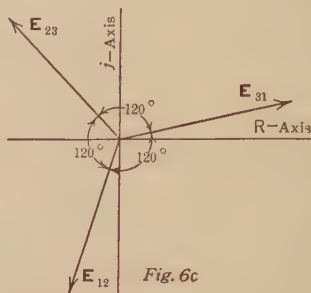


Fig. 6c

FIG. 6a. A three-phase Emf. source connected in star or Y.

FIG. 6b. Vector diagram of the phase- and line-voltages of Fig. 6a.

FIG. 6c. The line-voltages in Fig. 6a are equal and equally displaced from each other by 120° .

The similarity of the vector diagram (Fig. 6b) to the letter Y is responsible for calling this type of connection the Y-connection.

The student's attention is called to the fact that the subscripts in successive equations follow a cyclic order. Thus wherever the subscript 1 occurs in Eq. (5a) the subscripts 2 and 3 appear in Eqs. (5b) and (5c) respectively. This cyclic transformation of subscripts characterizes all the equations occurring in the chapters on polyphase circuits. It arises from the symmetry in the choice of notations. Through the cyclic transformation of subscripts the student is aided in writing polyphase equations and in checking them after they have been written. Its use is, therefore, very strongly urged.

(2) *The Three Line-Voltages* are measured between lines by connecting the voltmeter terminals to the points 1-2, 2-3, and 3-1 respectively. These voltages (Fig. 6b) are:

$$\left. \begin{aligned} \mathbf{E}_{12} &= \mathbf{E}_{02} - \mathbf{E}_{01} \quad (a) \\ \mathbf{E}_{23} &= \mathbf{E}_{03} - \mathbf{E}_{02} \quad (b) \\ \mathbf{E}_{31} &= \mathbf{E}_{01} - \mathbf{E}_{03} \quad (c) \end{aligned} \right\} \quad (6)$$

Since these line-voltages form a closed triangle their sum is zero. This can be easily proved by adding Eqs. (6), which gives:

$$\mathbf{E}_{12} + \mathbf{E}_{23} + \mathbf{E}_{31} = 0 \quad (7)$$

Not only are the phase-voltages of a three-phase balanced source equal and displaced from each other by 120° . The three line-voltages

are also equal and equally displaced from each other by 120° . This can be easily seen (Fig. 6c) by drawing vectors (equal to $\mathbf{E}_{12}$, $\mathbf{E}_{23}$ and $\mathbf{E}_{31}$), all of which start from the origin 0.

From the vector diagram, Fig. 6b, we can deduce a numerical relation between the effective value E_p of the phase-voltage and the effective value E_L of the line-voltage. Thus let a perpendicular be dropped from 0 (Fig. 6b) to $\mathbf{E}_{12}$. This perpendicular bisects the line $\mathbf{E}_{12}$ and the angle 102 because the triangle 102 is an isosceles triangle. We thus have:

$$\left. \begin{aligned} (1/2) E_{12} &= E_{01} \sin 60^\circ = E_{01}(\sqrt{3}/2) & (a) \\ \text{or } E_L &= \sqrt{3} E_p & (b) \end{aligned} \right\} \quad (8)$$

(b) *The Four-Phase Star-Connection.* — Let the terminals 1', 2', 3', and 4' of a four-phase Emf. source (Fig. 7a) be connected to a common point 0. There exist eight different voltages: (1) four phase-voltages, and (2) four line-voltages.

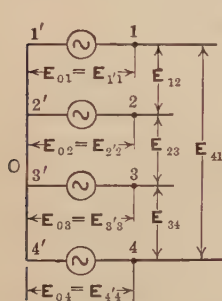


Fig. 7a

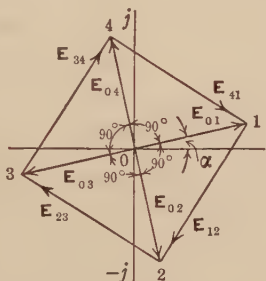


Fig. 7b

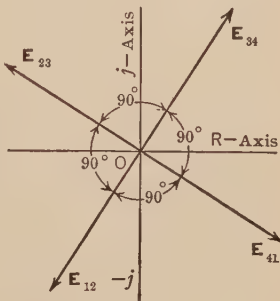


Fig. 7c

FIG. 7a. A four-phase Emf. source connected in star.

FIG. 7b. Vector diagram of the phase- and line-voltages of Fig. 7a.

FIG. 7c. The line-voltages in Fig. 7a are equal and equally displaced from each other by 90° .

(1) *The Four Phase-Voltages* are measured by connecting one terminal of a voltmeter to the point 0 and the other terminal to the points 1, 2, 3, and 4 successively. These voltages are identical (see Fig. 7a) with the terminal Emfs. of the four separate phases; or:

$$\left. \begin{aligned} \mathbf{E}_{01} &= \mathbf{E}_{1'1} & (a) & \quad \mathbf{E}_{02} = \mathbf{E}_{2'2} & (b) \\ \mathbf{E}_{03} &= \mathbf{E}_{3'3} & (c) & \quad \mathbf{E}_{04} = \mathbf{E}_{4'4} & (d) \end{aligned} \right\} \quad (9)$$

By assumption, the terminal Emfs. of the four phases are equal and equally displaced from each other. Hence the phase-voltages $\mathbf{E}_{01} \dots \mathbf{E}_{04}$ (Fig. 7b) are also equal and equally displaced from each other by 90° , and their sum is zero.

(2) *The Four Line-Voltages* are measured (across the lines) between the terminals 1-2, 2-3, 3-4, and 4-1 respectively. From the vector diagram Fig. 7b we have:

$$\left. \begin{aligned} \mathbf{E}_{12} &= \mathbf{E}_{02} - \mathbf{E}_{01} & (a) \\ \mathbf{E}_{34} &= \mathbf{E}_{04} - \mathbf{E}_{03} & (c) \end{aligned} \right\} \quad \left. \begin{aligned} \mathbf{E}_{23} &= \mathbf{E}_{03} - \mathbf{E}_{02} & (b) \\ \mathbf{E}_{41} &= \mathbf{E}_{01} - \mathbf{E}_{04} & (d) \end{aligned} \right\} \quad (10)$$

whence by addition:

$$\mathbf{E}_{12} + \mathbf{E}_{23} + \mathbf{E}_{34} + \mathbf{E}_{41} = 0 \quad (11)$$

That the line-voltages of a four-phase balanced source are equal and equally displaced from each other by 90° may be seen from Fig. 7c, which represents four vectors equal to the line vectors but so drawn that they all start from the origin of the R - j plane.

From the vector diagram (Fig. 7b) we can deduce a simple relation between the effective values of the phase-voltages E_p and the line-voltages E_L . Thus, taking any of the right triangles (say 102) we have:

$$\left. \begin{aligned} E_{12}^2 &= E_{01}^2 + E_{02}^2 & (a) \\ \text{or } E_L &= \sqrt{E_p^2 + E_p^2} = \sqrt{2} E_p & (b) \end{aligned} \right\} \quad (12)$$

6. Emf. Relations in a Mesh-Connected Source.—(a) *The Three-Phase Mesh- or Δ -Connection.*—It is shown, in paragraph 5a, above, that the sum of the terminal Emfs. of the phases of a balanced polyphase alternator is zero. Hence no harm could accrue from connecting the terminals of the separate windings in such a manner as to form a closed loop provided that care is exercised, in making these connections, so that the sum of the Emfs. around the loop is zero. Thus in Fig. 5b the Emfs. $\mathbf{E}_{1'1}$, $\mathbf{E}_{2'2}$ and $\mathbf{E}_{3'3}$ are equal and equally displaced from each other. Hence:

$$\mathbf{E}_{1'1} + \mathbf{E}_{2'2} + \mathbf{E}_{3'3} = 0 \quad (13)$$

This means that the pairs of terminals 1-2', 2-3', and 3-1' could be joined together to form a closed loop (Fig. 8a) and no current will circulate through that loop because the sum of the Emfs. around the loop is zero.

Since the points 1 and 2', 2 and 3', 3 and 1', are respectively at the same potential, the line-voltages $\mathbf{E}_{12}$, $\mathbf{E}_{23}$, and $\mathbf{E}_{31}$ are related to the phase-voltages in the following manner:

$$\left. \begin{aligned} \mathbf{E}_{12} &= \mathbf{E}_{2'2} & (a) \\ \mathbf{E}_{23} &= \mathbf{E}_{3'3} & (b) \\ \mathbf{E}_{31} &= \mathbf{E}_{1'1} & (c) \end{aligned} \right\} \quad (14)$$

These line-voltages are plotted in Figs. 8b and 8c. The latter figure emphasizes the fact that their sum is equal to zero and gives the reason

for the name of this connection. Indeed the similarity between the vector diagram Fig. 8c and the Greek letter delta (Δ) is responsible for calling this type of mesh-connection the delta-connection.

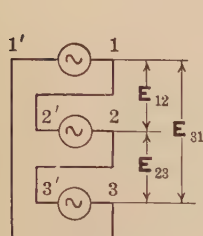


Fig. 8a

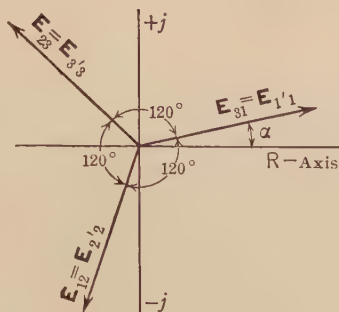


Fig. 8b

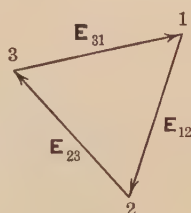


Fig. 8c

FIG. 8a. A three-phase Emf. source connected in mesh or delta.

FIG. 8b. Vector diagram of the phase- or line-voltages of Fig. 8a.

FIG. 8c. The vector voltages in Fig. 8b form a closed polygon resembling the Greek letter delta (Δ).

The equality between the line- and phase-voltages of a delta-connected source need hardly be emphasized. The student can readily perceive that a delta-connected source has only three voltages — the line- or the phase-voltages. This, of course, is not true of a *Y*-connected source where the line-voltages are different from the phase-voltages.

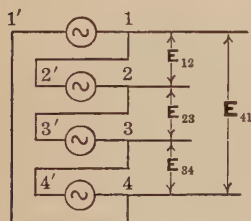


Fig. 9a

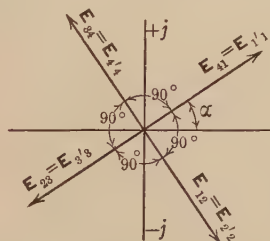


Fig. 9b

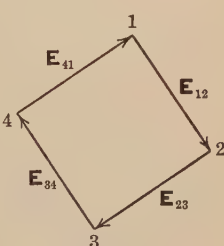


Fig. 9c

FIG. 9a. A four-phase Emf. source connected in mesh.

FIG. 9b. Vector diagram of the phase- or line-voltages of Fig. 9a.

FIG. 9c. The vectors Fig. 9b form a closed polygon. Hence their sum is zero.

(b) *The Four-Phase Mesh-Connection.* — It is shown in Art. 5b, above, that the sum of the terminal Emfs. of a four-phase balanced source is equal to zero. Hence it is possible to properly connect these terminals one to the other, thereby forming a closed loop as in Fig. 9a, without obtaining a circulating current in the loop. When the Emfs.

of a four-phase source are connected as in Fig. 9a, they are said to be mesh-connected. The pairs of terminals 1-2', 2-3', 3-4', and 4-1' (Fig. 9a) are respectively joined together. Hence they are at the same potential and:

$$\left. \begin{array}{ll} \mathbf{E}_{12} = \mathbf{E}_{2'2} & (a) \\ \mathbf{E}_{34} = \mathbf{E}_{4'4} & (c) \end{array} \right\} \quad \left. \begin{array}{ll} \mathbf{E}_{23} = \mathbf{E}_{3'3} & (b) \\ \mathbf{E}_{41} = \mathbf{E}_{1'1} & (d) \end{array} \right\} \quad (15)$$

The line-voltages $\mathbf{E}_{12} \dots \mathbf{E}_{41}$ are vectorially shown in Figs. 9b and 9c. The latter figure emphasizes the fact that the sum of the line-voltages is zero.

7. Methods of Loading Polyphase Sources. — A polyphase source is said to be loaded when it supplies current to some circuit characteristics (or load) connected to its terminals. The combination of a polyphase Emf. source and load is known as a polyphase circuit. Just as the several phases of the source may or may not be interconnected, so also the several branches of the load may be separate or connected in star or mesh. We thus have the types of polyphase circuits set forth in Table I.

TABLE I. TYPES OF POLYPHASE CIRCUITS

Type of circuit	Methods of connection		Name of circuit	Fig.
	Source	Load		
<i>A</i>	no interconnection	no interconnection	non-interconnected	10a
<i>B</i>	star	star	star-star	11a
<i>C</i>	mesh	star	mesh-star	12a
<i>D</i>	mesh	mesh	mesh-mesh	13a
<i>E</i>	star	mesh	star-mesh	14a

Again, just as the Emfs. of the source may or may not be balanced so the separate loads may or may not be identical. A polyphase circuit is said to be *balanced* when the source is balanced and the loads are identical. This chapter is limited exclusively to balanced circuits. Moreover, since the treatment of the four-phase balanced circuit does not differ materially from that of the three-phase circuit we shall limit the text to the three-phase circuit. The extension of the treatment to the four-phase circuit is reserved as an exercise to the reader.

8. Current and Voltage Relations in a Balanced Three-Phase Circuit of Type A (Non-interconnected). — Consider a three-phase source (Fig. 10a) with identical loads Z connected across each of the terminals of the three phases. We have three independent series circuits in which

the Emfs. and currents satisfy the relations (see Art. 3, Chapter VII):

$$\left. \begin{aligned} \mathbf{E}_{1'1} &= \mathbf{Z}\mathbf{I}_{1'1} & (a) \\ \mathbf{E}_{2'2} &= \mathbf{Z}\mathbf{I}_{2'2} & (b) \\ \mathbf{E}_{3'3} &= \mathbf{Z}\mathbf{I}_{3'3} & (c) \end{aligned} \right\} \quad (16)$$

Here are three independent vector equations any of which can be solved if two of the quantities (for instance, $\mathbf{E}_{1'1}$ and $\mathbf{Z}$) are known. The vector relation between the phase Emfs. and currents is shown in Fig. 10b. Observe that the phase-voltages are equal and the phase-currents are equal and equally displaced from each other by 120° .

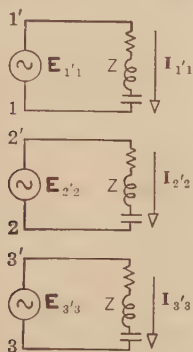


Fig. 10a

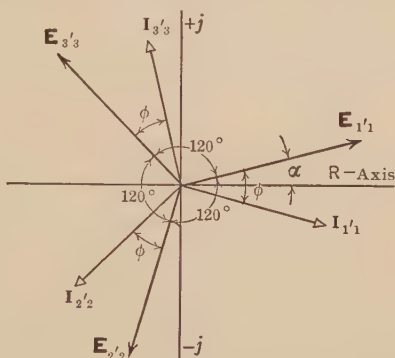


Fig. 10b

FIG. 10a. A non-interconnected three-phase balanced circuit.

FIG. 10b. Vector diagram of Emfs. and currents in the polyphase circuit Fig. 10a.

9. Current and Voltage Relations in a Three-Phase Balanced Circuit of Type B (Star-Star).— Fig. 11a shows a three-phase balanced circuit having a star-connected Emf. source and a star-connected load. We shall designate the source neutral by the letter 0 and the load neutral by N . The phase-voltages of the source and the load-impedances are given and it is required to determine the line- (or phase-currents).

Call $\mathbf{E}_{0N}$ the difference of potential between the neutrals 0 and N . Applying Kirchhoff's Emf. law to the circuits 01N0, 02N0, and 03N0 we have:

$$\left. \begin{aligned} \mathbf{E}_{01} - \mathbf{E}_{0N} &= \mathbf{Z}\mathbf{I}_{01} & (a) \\ \mathbf{E}_{02} - \mathbf{E}_{0N} &= \mathbf{Z}\mathbf{I}_{02} & (b) \\ \mathbf{E}_{03} - \mathbf{E}_{0N} &= \mathbf{Z}\mathbf{I}_{03} & (c) \end{aligned} \right\} \quad (17)$$

Applying Kirchhoff's current law to the point N we have:

$$\mathbf{I}_{01} + \mathbf{I}_{02} + \mathbf{I}_{03} = 0 \quad (18)$$

Adding Eqs. (17) we obtain:

$$(\mathbf{E}_{01} + \mathbf{E}_{02} + \mathbf{E}_{03}) - 3 \mathbf{E}_{0N} = (\mathbf{I}_{01} + \mathbf{I}_{02} + \mathbf{I}_{03})Z \quad (19)$$

But the sum of the three star-voltages is zero because they are equal and equally displaced. Moreover, by Eq. (18), the sum of the line-currents is zero. Thus Eq. (19) reduces to:

$$3 \mathbf{E}_{0N} = 0 \quad \text{or} \quad \mathbf{E}_{0N} = 0 \quad (20)$$

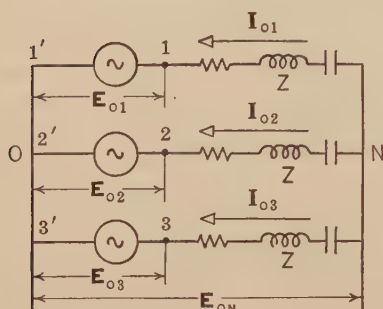


Fig. 11a

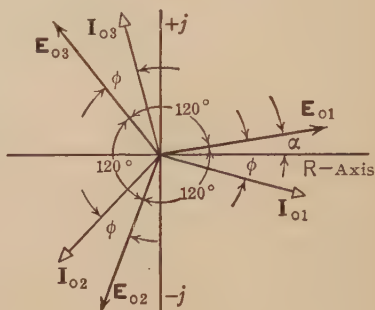


Fig. 11b

FIG. 11a. A star-star, three-phase, balanced circuit.

FIG. 11b. Vector diagram of Emfs. and currents in the polyphase circuit Fig. 11a.

Substituting $\mathbf{E}_{0N} = 0$ in Eqs. (17) we have:

$$\left. \begin{aligned} \mathbf{E}_{01} &= Z\mathbf{I}_{01} & (a) \\ \mathbf{E}_{02} &= Z\mathbf{I}_{02} & (b) \\ \mathbf{E}_{03} &= Z\mathbf{I}_{03} & (c) \end{aligned} \right\} \quad (21)$$

We thus conclude that in a balanced star-star circuit each phase may be treated as an independent series circuit having an impedance Z equal to the phase impedance and an Emf. $\mathbf{E}$ equal to the phase Emf. Use of this fact is often made in the computation of the performance of three-phase circuits, particularly for long distance transmission.

The vector relations between the phase Emfs. and phase- (or line-currents) is given by the vector diagram Fig. 11b.

10. Current and Voltage Relations in a Balanced Three-Phase Circuit of Type C (Mesh-Star).— Consider the balanced three-phase circuit shown in Fig. 12a and let it be required to develop expressions for the relations between the line- (or phase-) Emfs. and the line-currents.

By Kirchhoff's Emf. law we have:

$$\left. \begin{aligned} \mathbf{E}_{12} &= Z(\mathbf{I}_{N2} - \mathbf{I}_{N1}) & (a) \\ \mathbf{E}_{23} &= Z(\mathbf{I}_{N3} - \mathbf{I}_{N2}) & (b) \end{aligned} \right\} \quad (22)$$

Applying Kirchhoff's current law to the point N we obtain:

$$I_{N1} + I_{N2} + I_{N3} = 0 \quad (23)$$

Solving Eq. (22a) for I_{N1} and Eq. (22b) for I_{N3} , substituting these values in Eq. (23), and simplifying we obtain:

$$I_{N2} = (E_{12} - E_{23})/3Z \quad (24)$$

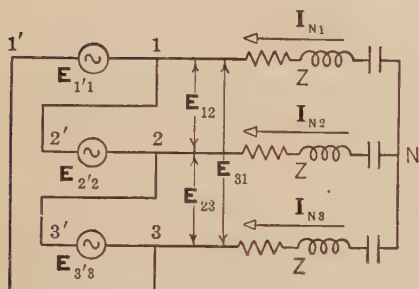


Fig. 12a

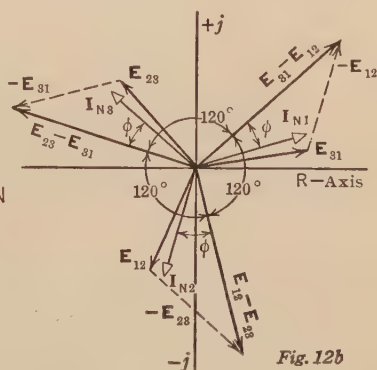


Fig. 12b

Fig. 12a. A mesh-star, three-phase, balanced circuit.

Fig. 12b. Vector diagram of Emfs. and currents in the polyphase circuit Fig. 12a.

The values of I_{N1} and I_{N3} can now be written either by substituting Eq. (24) in Eqs. (22) or by using the cyclic transformation of subscripts. We thus have:

$$\left. \begin{aligned} I_{N1} &= (E_{31} - E_{12})/3Z & (a) \\ I_{N2} &= (E_{12} - E_{23})/3Z & (b) \\ I_{N3} &= (E_{23} - E_{31})/3Z & (c) \end{aligned} \right\} \quad (25)$$

Eqs. (25) state that the Emf. across any load impedance is in phase with the vector difference between the line Emfs. and has an effective value equal to one third of this vector difference.

This deduction is confirmed by the vector diagram (Fig. 12b) which shows the line-voltages and currents in this type of circuit. The vector line-currents lag the vector difference of the line-voltages (*not the line-voltages themselves*) by angles ϕ .

11. Current and Voltage Relations in a Three-Phase Balanced Circuit of Type D (Mesh-Mesh). — Let a delta-connected source (Fig. 13a) supply power to a delta-connected load. Let the line-voltages and load-impedances Z_L be known. It is required to find the line- and load-currents.

We have by Kirchhoff's Emf. law (see Fig. 13a):

$$\left. \begin{aligned} \mathbf{E}_{2'2} = \mathbf{E}_{12} &= \mathbf{Z}\mathbf{I}_{12} \quad (a) \\ \mathbf{E}_{3'3} = \mathbf{E}_{23} &= \mathbf{Z}\mathbf{I}_{23} \quad (b) \\ \mathbf{E}_{1'1} = \mathbf{E}_{31} &= \mathbf{Z}\mathbf{I}_{31} \quad (c) \end{aligned} \right\} \quad (26)$$

Knowing $\mathbf{E}$ and $\mathbf{Z}$ the load-currents can be easily determined.

The line-currents $\mathbf{I}_{11}$, $\mathbf{I}_{22}$ and $\mathbf{I}_{33}$ may be evaluated by applying Kirchhoff's current law to the points 1, 2, and 3 (Fig. 13a). This gives:

$$\left. \begin{aligned} \mathbf{I}_{11} &= \mathbf{I}_{31} - \mathbf{I}_{12} \quad (a) \\ \mathbf{I}_{22} &= \mathbf{I}_{12} - \mathbf{I}_{23} \quad (b) \\ \mathbf{I}_{33} &= \mathbf{I}_{23} - \mathbf{I}_{31} \quad (c) \end{aligned} \right\} \quad (27)$$

We thus have six simultaneous vector equations from which the three line-currents and the three load-currents may be determined.*

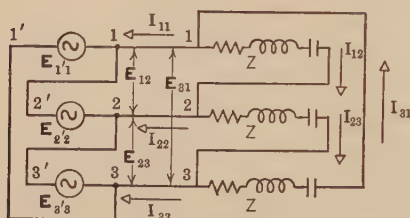


Fig. 13a

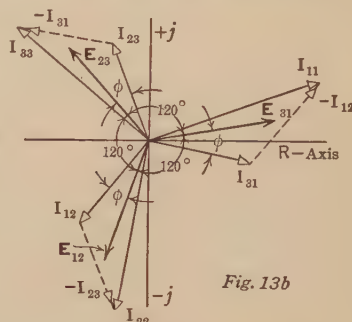


Fig. 13b

FIG. 13a. A mesh-mesh, three-phase, balanced circuit.

FIG. 13b. Vector diagram of the Emfs. and currents in the polyphase circuit Fig. 13a.

An interesting relation exists between the effective value of the line-currents I_L and phase-currents I_p in this type of balanced three-phase circuits. This relation may be deduced from the vector diagram (Fig. 13b) by dropping a line perpendicular to $\mathbf{I}_{11}$ from the terminus of $\mathbf{I}_{31}$. We have:

$$\left. \begin{aligned} (1/2)\mathbf{I}_{11} &= \mathbf{I}_{31} \sin 60^\circ = \mathbf{I}_{31} \sqrt{3}/2 \quad (a) \\ \text{or} \quad \mathbf{I}_L &= \sqrt{3} \mathbf{I}_p \quad (b) \end{aligned} \right\} \quad (28)$$

* By assumption, the circuit is balanced. Hence the load-currents are equal and equally displaced from each other by 120° . The same holds true also of the line-currents. It is therefore only necessary to compute one load-current; the other load-currents may be obtained by merely shifting the phase position of that current through 120° . Again it is only necessary to evaluate one of the line-currents by the use of Eqs. (27). The other line-currents can be obtained by adding or subtracting 120° and 240° respectively to the phase angle of the current so evaluated.

12. Current and Voltage Relations in a Balanced Three-Phase Circuit of Type E (Star-Mesh). — Fig. 14a shows a star-connected balanced Emf. source supplying power to a delta-connected balanced load. The phase-voltages E_{01} , E_{02} and E_{03} as well as the load-impedances Z are known. It is required to compute the phase- and line-currents.

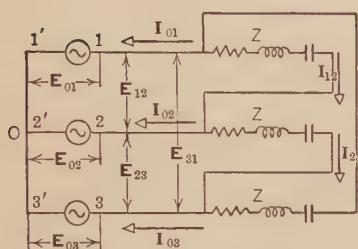


Fig. 14a

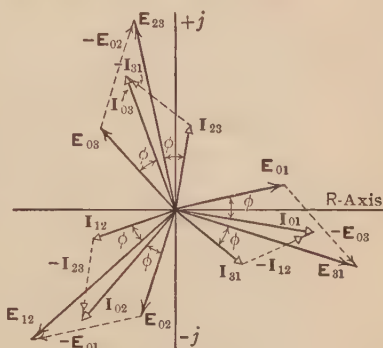


Fig. 14b

FIG. 14a. A star-mesh, three-phase, balanced circuit.

FIG. 14b. Vector diagram of the Emfs. and currents in the circuit Fig. 14a.

Solution. — Compute the line-voltages by the use of Eqs. (6). This case then reduces to the mesh-mesh case discussed in Art. 11 above. Thus by Kirchhoff's Emf. law :

$$\left. \begin{aligned} E_{01} - E_{02} &= E_{12} = ZI_{12} & (a) \\ E_{02} - E_{03} &= E_{23} = ZI_{23} & (b) \\ E_{03} - E_{01} &= E_{31} = ZI_{31} & (c) \end{aligned} \right\} \quad (29)$$

The line-currents I_{01} , I_{02} and I_{03} may be evaluated by applying Kirchhoff's current law to the points 1, 2, and 3 (Fig. 14b); or:

$$\left. \begin{aligned} I_{01} &= I_{31} - I_{12} & (a) \\ I_{02} &= I_{12} - I_{23} & (b) \\ I_{03} &= I_{23} - I_{31} & (c) \end{aligned} \right\} \quad (30)$$

The similarity between Eqs. (26) and (29) and between Eqs. (27) and (30) is apparent.

Eqs. (29) and (30) give six vector equations from which the three line-currents and the three load-currents (Fig. 14) may be computed.

13. Equivalent Mesh and Star Three-Phase Circuits. — Referring to the four cases of interconnected three-phase circuits, it will be observed that the simplest is the star-star circuit. Besides its simplicity, this circuit possesses the further advantage that each phase may be treated as an independent series circuit having an impedance Z equal

to the phase impedance and an Emf. $\mathbf{E}$ equal to the phase Emf. Thus it becomes desirable, in some instances, to convert the other types of polyphase circuits into an equivalent star-star circuit.

Two polyphase circuits are said to be equivalent when:

- (1) The vector line-voltages of the two circuits are identical (are equal in magnitude and phase position).
- (2) The vector line-currents are identical (are equal in magnitude and phase position).

The problem of converting a balanced three-phase circuit of type C , D , or E to an equivalent star-star circuit consists, therefore, in determining one or both of the following:

- (a) A star-connected source whose line-voltages are identical with the given line- (phase-) voltages of the mesh source.
- (b) A star-connected load which draws the identical vector line-currents as a mesh load.

Table I shows that:

- (1) Circuits of type C have a star load. Hence only the source need be converted to an equivalent star source in order to treat this circuit as a star-star circuit.
- (2) Circuits of type E have a star source. Hence if the mesh load is converted to an equivalent star load the circuit becomes a star-star circuit.
- (3) Circuits of type D have both the source and load mesh connected. Consequently both the source and load should be converted to equivalent star in order to change the circuit to a star-star circuit.

We shall now indicate how a mesh source and a mesh load may be replaced by an equivalent star source or an equivalent star load respectively.

A. The Equivalent Star Source. — In order to find a balanced set of star voltages that is equivalent to a balanced set of line- or mesh-voltages we shall make use of the vector relations, between the star- and line-voltages, shown in Fig. 6b. Observe that:

- (1) The effective value of the star-voltage is equal to $(1/\sqrt{3})$ times the effective value of the line-voltages.
- (2) The star-voltage $\mathbf{E}_{01}$ leads the line-voltage $\mathbf{E}_{12}$ by 150° .

Hence if the line-voltages are given as:

$$\left. \begin{aligned} \mathbf{E}_{12} &= E_L \angle \alpha & (a) \\ \mathbf{E}_{23} &= E_L \angle (\alpha - 120^\circ) & (b) \\ \mathbf{E}_{31} &= E_L \angle (\alpha + 120^\circ) & (c) \end{aligned} \right\} \quad (31)$$

The equivalent star-voltages would be:

$$\left. \begin{aligned} \mathbf{E}_{01} &= (E_L/\sqrt{3}) \angle (\alpha + 150^\circ) & (a) \\ \mathbf{E}_{02} &= (E_L/\sqrt{3}) \angle (\alpha + 30^\circ) & (b) \\ \mathbf{E}_{03} &= (E_L/\sqrt{3}) \angle (\alpha + 270^\circ) & (c) \end{aligned} \right\} \quad (32)$$

B. The Equivalent Star Load.—In order to convert the mesh-connected load to an equivalent star load, it is necessary to compute the line-currents for both types of connection, equate them and solve for the mesh impedance in terms of the star impedance. Thus the line-currents for a star-star circuit are (see Eqs. 21):

$$\left. \begin{aligned} \mathbf{I}_{01} &= \mathbf{E}_{01}/Z_s & (a) \\ \mathbf{I}_{02} &= \mathbf{E}_{02}/Z_s & (b) \\ \mathbf{I}_{03} &= \mathbf{E}_{03}/Z_s & (c) \end{aligned} \right\} \quad (33)$$

where Z_s stands for a star impedance.

The line-currents of a mesh-mesh circuit are (see Eqs. 26 and 27):

$$\left. \begin{aligned} \mathbf{I}_{11} &= \mathbf{I}_{31} - \mathbf{I}_{12} = (\mathbf{E}_{31} - \mathbf{E}_{12})/Z_m & (a) \\ \mathbf{I}_{22} &= \mathbf{I}_{12} - \mathbf{I}_{23} = (\mathbf{E}_{12} - \mathbf{E}_{23})/Z_m & (b) \\ \mathbf{I}_{33} &= \mathbf{I}_{23} - \mathbf{I}_{31} = (\mathbf{E}_{23} - \mathbf{E}_{31})/Z_m & (c) \end{aligned} \right\} \quad (34)$$

where Z_m stands for a mesh impedance.

But the circuits are equivalent if the line-currents as expressed by Eqs. (33) and (34) are identical. Hence:

$$\left. \begin{aligned} (\mathbf{E}_{01}/Z_s) &= (\mathbf{E}_{31} - \mathbf{E}_{12})/Z_m & (a) \\ (\mathbf{E}_{02}/Z_s) &= (\mathbf{E}_{12} - \mathbf{E}_{23})/Z_m & (b) \\ (\mathbf{E}_{03}/Z_s) &= (\mathbf{E}_{23} - \mathbf{E}_{31})/Z_m & (c) \end{aligned} \right\} \quad (35)$$

Substituting Eq. (6) in Eqs. (35) and simplifying we obtain:

$$\left. \begin{aligned} (\mathbf{E}_{01}/Z_s) &= [3 \mathbf{E}_{01} - (\mathbf{E}_{01} + \mathbf{E}_{02} + \mathbf{E}_{03})]/Z_m & (a) \\ (\mathbf{E}_{02}/Z_s) &= [3 \mathbf{E}_{02} - (\mathbf{E}_{01} + \mathbf{E}_{02} + \mathbf{E}_{03})]/Z_m & (b) \\ (\mathbf{E}_{03}/Z_s) &= [3 \mathbf{E}_{03} - (\mathbf{E}_{01} + \mathbf{E}_{02} + \mathbf{E}_{03})]/Z_m & (c) \end{aligned} \right\} \quad (36)$$

But $(\mathbf{E}_{01} + \mathbf{E}_{02} + \mathbf{E}_{03}) = 0$ therefore Eqs. (36) reduce to the form:

$$Z_s = (Z_m/3) \quad \text{or} \quad Z_m = 3 Z_s \quad (37)$$

or the star-connected load that is equivalent to a mesh-connected load must have a complex impedance equal to 1/3 that of the mesh load.

14. Harmonics in Three-Phase Balanced Sources.—Certain very interesting phenomena arise in connection with harmonics occurring in three-phase balanced sources. For convenience, in discussion, we shall divide the odd harmonics (no even harmonics occur in electrical machinery) into two classes:

- (a) *The Triplen Harmonics.* — These include all odd harmonics which are integer multiples of three. The order of these harmonics may be expressed by the relation $u = (6n + 3)$ where n is zero or any integer.
- (b) *The Non-Triplen Harmonics.* — These comprise all odd harmonics that are not integer multiples of three. The order v of any one of these harmonics may be expressed by the equation $v = (6n \pm 1)$.

(a) *The Triplen Harmonics.* — Let each of the non-sine three-phase balanced sources consist of a fundamental and odd harmonics, all of which are multiples of 3. The induced Emfs. are displaced from each other, in time, by 120° . Hence their instantaneous values are:

$$\left. \begin{aligned} e_{1'1} &= E_{m1} \sin(\omega_1 t + \alpha_1) + \sum E_{mu} \sin u(\omega_1 t + \alpha_u) & (a) \\ e_{2'2} &= E_{m1} \sin(\omega_1 t + \alpha_1 - 120^\circ) + \sum E_{mu} \sin u(\omega_1 t + \alpha_u - 120^\circ) & (b) \\ e_{3'3} &= E_{m1} \sin(\omega_1 t + \alpha_1 + 120^\circ) + \sum E_{mu} \sin u(\omega_1 t + \alpha_u + 120^\circ) & (c) \end{aligned} \right\} \quad (38)$$

But $u \times 120^\circ = (6n + 3) \times 120^\circ = \text{a multiple of } 360^\circ = 0^\circ$. Hence Eqs. (38) become:

$$\left. \begin{aligned} e_{1'1} &= E_{m1} \sin(\omega_1 t + \alpha_1) + \sum E_{mu} \sin u(\omega_1 t + \alpha_n) & (a) \\ e_{2'2} &= E_{m1} \sin(\omega_1 t + \alpha_1 - 120^\circ) + \sum E_{mu} \sin u(\omega_1 t + \alpha_n) & (b) \\ e_{3'3} &= E_{m1} \sin(\omega_1 t + \alpha_1 + 120^\circ) + \sum E_{mu} \sin u(\omega_1 t + \alpha_n) & (c) \end{aligned} \right\} \quad (39)$$

We conclude, therefore, that a triplen harmonic in the second and third phases is not displaced from the corresponding harmonic in the first phase by 120° but is actually in phase with it. For instance, the third harmonic in phase 1 is in phase with the third harmonic in phase 2 and with the third harmonic in phase 3.

Now let the three phases be connected in star and let each of the phase Emfs. contain triplen harmonic. The line-voltages (see Eqs. 6) are:

$$\left. \begin{aligned} e_{12} &= e_{2'2} - e_{1'1} & (a) \\ e_{23} &= e_{3'3} - e_{2'2} & (b) \\ e_{31} &= e_{1'1} - e_{3'3} & (c) \end{aligned} \right\} \quad (40)$$

Substituting the values of the phase voltages from Eqs. (39) in Eqs. (40) and simplifying we obtain:

$$\left. \begin{aligned} e_{12} &= E_{m1} \sin(\omega_1 t + \alpha_1 - 120^\circ) - E_{m1} \sin(\omega_1 t + \alpha_1) & (a) \\ e_{23} &= E_{m1} \sin(\omega_1 t + \alpha_1 + 120^\circ) - E_{m1} \sin(\omega_1 t + \alpha_1 - 120^\circ) & (b) \\ e_{31} &= E_{m1} \sin(\omega_1 t + \alpha_1) - E_{m1} \sin(\omega_1 t + \alpha_1 + 120^\circ) & (c) \end{aligned} \right\} \quad (41)$$

*This relation holds true in so far as the trigonometric (or circular) functions are concerned. Such is the case under consideration because we are dealing with the sine of the angle.

Eqs. (41) show that even though the triplen harmonics may occur in the balanced phase-voltages, yet if these voltages are connected in star such harmonics are absent from the line-voltages.

(b) *The Non-Triplen Harmonics.*— Let a three-phase balanced source consist of fundamentals and non-triplen odd harmonics. The instantaneous values of the Emfs. in each of the three phases are:

$$\left. \begin{aligned} e_{1'1} &= E_{m1} \sin (\omega_1 t + \alpha_1) & + \sum E_{mv} \sin v(\omega_1 t + \alpha_v) & (a) \\ e_{2'2} &= E_{m1} \sin (\omega_1 t + \alpha_1 - 120^\circ) + \sum E_{mv} \sin v(\omega_1 t + \alpha_v - 120^\circ) & (b) \\ e_{3'3} &= E_{m1} \sin (\omega_1 t + \alpha_1 + 120^\circ) + \sum E_{mv} \sin v(\omega_1 t + \alpha_v + 120^\circ) & (c) \end{aligned} \right\} \quad (42)$$

where $v = (6n \pm 1) =$ any odd integer which is not a multiple of 3.

Two types of non-triplen harmonics need be considered:

- (1) Those whose order may be represented by $v' = 6n + 1$ (for example, the 1st harmonic, the 7th, the 13th, etc.).
- (2) Those whose order may be represented by $v'' = 6n - 1$ (the 5th harmonic, the 11th, the 17th, etc.).

(1) Let the order of the harmonic be represented by $v' = (6n + 1)$ and let this value be substituted in Eqs. (42). Since $(6n + 1) 120^\circ = 120^\circ$ we have:

$$\left. \begin{aligned} e_{1'1} &= E_{m1} \sin (\omega_1 t + \alpha_1) & + \sum E_{mv'} \sin (v' \omega_1 t + v' \alpha_v) & (a) \\ e_{2'2} &= E_{m1} \sin (\omega_1 t + \alpha_1 - 120^\circ) + \sum E_{mv'} \sin (v' \omega_1 t + v' \alpha_v - 120^\circ) & (b) \\ e_{3'3} &= E_{m1} \sin (\omega_1 t + \alpha_1 + 120^\circ) + \sum E_{mv'} \sin (v' \omega_1 t + v' \alpha_v + 120^\circ) & (c) \end{aligned} \right\} \quad (43)$$

We thus conclude that non-triplen harmonics whose order is of the form $v' = 6n + 1$ are displaced from each other by 120° , exactly as is the fundamental. Moreover, this displacement is measured to the scale of that harmonic.

(2) Next let the order of the harmonic be represented by $v'' = (6n - 1)$ and let this value be substituted for v in Eqs. (42). Since $(6n - 1) 120^\circ = -120^\circ$ and $(6n - 1) (-120^\circ) = +120^\circ$ we have:

$$\left. \begin{aligned} e_{1'1} &= E_{m1} \sin (\omega_1 t + \alpha_1) & + \sum E_{mv''} \sin (v'' \omega_1 t + v'' \alpha_v) & (a) \\ e_{2'2} &= E_{m1} \sin (\omega_1 t + \alpha_1 - 120^\circ) + \sum E_{mv''} \sin (v'' \omega_1 t + v'' \alpha_v + 120^\circ) & (b) \\ e_{3'3} &= E_{m1} \sin (\omega_1 t + \alpha_1 + 120^\circ) + \sum E_{mv''} \sin (v'' \omega_1 t + v'' \alpha_v - 120^\circ) & (c) \end{aligned} \right\} \quad (44)$$

Eqs. (44) establish the fact that non-triplen harmonics whose order is of the form $v'' = (6n - 1)$ are so displaced that a harmonic in the third phase lags that in the first by 120° and the same harmonic in the second phase lags that in the first by 240° or leads it by 120° measured to the scale of that harmonic.

For purposes of terminology we shall call *positive phase sequence* such a sequence of the phases that phase 1 leads phase 2 by 120° and leads phase 3 by 240° . Such a sequence occurs, for example, in the case of the fundamental and the non-triplen harmonics whose order is $v' = (6n + 1)$. We shall call a *negative phase sequence* such a sequence of the phases that phase 1 leads phase 3 by 120° and leads phase 2 by 240° . Example of this type of sequence is offered by non-triplen harmonics whose order is $v'' = 6n - 1$.

15. Current-Voltage Relations in Three-Phase Balanced Circuits with Non-Sine Impressed Emfs. — Consider a three-phase balanced source in which the Emfs. are non-sinusoidal.

Let it be required to determine the currents flowing through a balanced load of impedances Z_n when the source and load are so connected as to give circuits of types *A*, *B*, *C*, *D*, and *E* respectively. The problem naturally subdivides itself into as many parts as there are types of circuits. Moreover the triplen harmonics (see Art. 12) in the three phases are in phase but the non-triplen harmonics are 120° out of phase. Hence the behavior of polyphase circuits differs for the two general classes of odd harmonics. It will therefore be necessary to study the current-Emf. relations in the five types of polyphase circuits for both the non-triplen and the triplen harmonics.

A. THE NON-TRIPLen HARMONICS. — The current-Emf. relations of the non-triplen harmonics do not differ essentially from the current-voltage relations of the fundamental. Indeed for harmonics whose order is $v' = (6n + 1)$ the relations are identical with those developed for the fundamental. For harmonics whose order is $v'' = (6n - 1)$ the relations would be identical provided phases 2 and 3 are interchanged.* We can therefore dismiss the question of the non-triplen harmonics with the statement that these harmonics possess properties which are identical with those of the fundamental and hence the current-voltage relations in circuits comprising such harmonics may be obtained by resorting to the treatment given in Arts. 8 to 13. It is to be understood that each harmonic acts as though it were alone sending its own current through the polyphase circuit and that the instantaneous value of the total current through the circuit is the sum of the instantaneous currents due to all the harmonics.

B. THE TRIPLen HARMONICS. — We shall discuss the relations of the triplen Emfs. and currents in all the types of polyphase circuits:

(a) *Triplen Harmonics in Circuits of Type A (Non-Interconnected).* — Referring to Fig. 10a, let the Emfs. consist exclusively of triplen har-

* The necessity of interchanging phases 2 and 3 arises from the fact that this class of harmonics possesses a negative phase sequence (see Art. 12 above).

monics of any order u such that the triplen harmonic Emf. in the three phases (see Eqs. 39) are:

$$e_{1'1} = e_{2'2} = e_{3'3} = \sum E_{mu} \sin u(\omega_1 t + \alpha_u) \quad (45)$$

Let the complex impedances of the circuits to the u th harmonic be Z_u . Then the triplen harmonic currents in the three phases are:

$$i_{1'1} = i_{2'2} = i_{3'3} = \sum (E_{mu}/Z_u) \sin u(\omega_1 t + \alpha_u - \phi_u/u) \quad (46)$$

(b) *Triplen Harmonics in Circuits of Type B (Star-Star)*. — Two cases arise when a three-phase circuit of the type shown in Fig. 11a is subjected to a polyphase source comprising triplen harmonics:

Case (1) The Neutrals N and O Are Isolated. That is, they are not connected as shown in Fig. 11a.

Case (2) The Neutrals N and O Are Connected Together by an Impedance Z.

Of course Case 1 can be deduced from Case 2 by making Z infinite. For pedagogical reasons, however, it is best to consider the two cases separately, assuming, of course, in Case 2 that Z is always finite.

Case (1). Neutrals Isolated. — Consider the circuit, Fig. 11a. Let i_{01} , i_{02} and i_{03} be the instantaneous line currents due to the phase Emfs. e_{01} , e_{02} and e_{03} of the u th triplen harmonic. Let e_{0N} be the potential difference between the neutrals O and N and let Z_u be the load impedance for the u th harmonic. By Kirchhoff's Emf. law we have:

$$\left. \begin{aligned} (e_{01} - e_{0N}) &= i_{01} Z_u & (a) \\ (e_{02} - e_{0N}) &= i_{02} Z_u & (b) \\ (e_{03} - e_{0N}) &= i_{03} Z_u & (c) \end{aligned} \right\} \quad (47)$$

By Kirchhoff's current law the sum of the instantaneous values of the currents at point N is zero; or:

$$i_{01} + i_{02} + i_{03} = 0 \quad (48)$$

Adding Eqs. (47) and substituting zero for the sum of the currents we obtain:

$$\left. \begin{aligned} (e_{01} + e_{02} + e_{03}) - 3 e_{0N} &= 0 & (a) \\ 3 e_{01} &= 3 e_{0N} & (b) \end{aligned} \right\} \quad (49)$$

or (see Eq. 45)

If the source consists of several triplen harmonics then the cumulative effect of all the harmonics is:

$$e'_{0N} = \sum e_{0N} = \sum e_{01} = \sum E_{mu} \sin u(\omega_1 t + \alpha_u) \quad (50)$$

Eq. (50) states that in a three-phase star-star balanced circuit (Fig. 11a) the effect of the triplen harmonics is to create an Emf. e'_{0N} between

the points 0 and N (that is, between the source and load neutrals) of an instantaneous value equal to the sum of the instantaneous values of all the triplen harmonics in any one phase. In other words, if the terminal 0 (Fig. 11a) is grounded and a voltmeter is connected between the ground and the neutral N , the voltmeter will read such a voltage E'_{0N} that:

$$E'_{0N} = [E_3^2 + E_9^2 + \dots + E_u^2]^{1/2} \quad (51)$$

On the other hand, an oscillograph connected between the points 0 and N (Fig. 11a) gives a record e'_{0N} which comprises nothing but triplen harmonics. On account of the fact that the instantaneous value of the Emf. e'_{0N} varies between zero and positive and negative maxima, the neutral N is commonly known as the *oscillating neutral*.

Since $e_{0N} = e_{01} = e_{02} = e_{03}$, the difference between any phase-voltage and the neutral voltage (Eq. 47) is zero. Hence there exist no triplen harmonic currents in the lines (Fig. 11a).

Case (2). Neutrals Connected. — Now let the neutrals N and 0 (Fig. 11a) be connected by an impedance Z'_u and let the equivalent load-impedance to the u th harmonic be Z_u . Then by Kirchhoff's Emf. law:

$$\left. \begin{aligned} e_{01} &= (Z_u + Z'_u)i_{01} & (a) \\ e_{02} &= (Z_u + Z'_u)i_{02} & (b) \\ e_{03} &= (Z_u + Z'_u)i_{03} & (c) \end{aligned} \right\} \quad (52)$$

and by Kirchhoff's current law:

$$i_{0N} = i_{01} + i_{02} + i_{03} = 3 i_{01} \quad (53)$$

When the phase Emfs. comprise several triplen harmonics the currents due to these harmonics are:

$$\left. \begin{aligned} i'_{01} = i'_{02} = i'_{03} &= \sum \left\{ E_{mu} / (Z_u + Z'_u) \right\} \sin u(\omega_1 t + \alpha_u - \phi_u / u) & (a) \\ \text{and} & i'_{0N} = 3 i'_{01} & (b) \end{aligned} \right\} \quad (54)$$

Thus when the neutrals N and 0 (Fig. 11a) are connected, triplen harmonic currents, of magnitudes given by Eq. (54a), flow through each of the lines, and the three currents return through the neutral impedance Z'_u . Hence the neutral impedance carries a triplen harmonic current i_{0N} (see Eq. 54b) equal to the sum of all the triplen harmonic currents in the three lines.

(c, d) *Triplen Harmonics in Circuits of Types C and D.* — It is shown in Art. 14 that the line-voltages are free from triplen harmonics. Hence there exist no triplen harmonic currents in the loads, Figs. 12a and 13a.

In each of these two types of circuits the triplen harmonic Emfs. form circulating currents which heat up the winding of the Emf. source. It is extremely important, therefore, before using this type of connection,

to make sure that either no triplen harmonics exist in the Emf. source or, if they do exist, that their magnitudes are such that no excessive heating results.

The effective value of the Emf. acting on the delta source is equal to the sum of the effective values of the triplen Emfs. in the three phases; or:

$$E = 3 [E_3^2 + E_9^2 + \dots + E_n^2]^{1/2} \quad (55)$$

The circulating current is limited solely by the sum of the impedances of the three phases. Hence the circulating current is:

$$I = E/3 Z_p = [E_3^2 + E_9^2 + \dots + E_n^2]^{1/2} / Z_p \quad (56)$$

where Z_p is the equivalent impedance per phase of the armature winding of the machine.

(e) *Triplen Harmonics in Circuits of Type E.*—It is shown in Art. 14 that the line-voltages are free from triplen harmonics. Hence there exist no triplen harmonic currents in the load, Fig. 14a.

C. TOTAL CURRENT DUE TO ALL THE HARMONICS.—Assume that the phase- and line-currents due to each harmonic Emf. have been determined as outlined above. The total line- or phase-current will have an instantaneous value equal to the sum of the instantaneous values of all the harmonic currents per line or phase respectively and will have an effective value equal to the square root of the sum of the squares of the effective values of the harmonic currents.

The solution of problem 15a is given (p. 255) in an effort to make this treatment clearer. The student will profit by studying this problem.

16. Power in Three-Phase Balanced Circuits with Sine Emfs.—

(a) *Instantaneous Power.*—Let the Emfs. across the three loads of a balanced three-phase circuit, of any type, be:

$$\left. \begin{aligned} e_{p1} &= E_m \sin (\omega t + \alpha) & (a) \\ e_{p2} &= E_m \sin (\omega t + \alpha - 120^\circ) & (b) \\ e_{p3} &= E_m \sin (\omega t + \alpha + 120^\circ) & (c) \end{aligned} \right\} \quad (57)$$

Let the currents flowing through the load due to these Emfs. be:

$$\left. \begin{aligned} i_{p1} &= I_m \sin (\omega t + \alpha \pm \phi) & (a) \\ i_{p2} &= I_m \sin (\omega t + \alpha - 120^\circ \pm \phi) & (b) \\ i_{p3} &= I_m \sin (\omega t + \alpha + 120^\circ \pm \phi) & (c) \end{aligned} \right\} \quad (58)$$

The instantaneous power in the three phases is:

$$\left. \begin{aligned} p_1 &= e_{p1} i_{p1} = (E_m I_m / 2) [\cos \phi - \cos 2 (\omega t + \alpha \pm \phi / 2)] & (a) \\ p_2 &= e_{p2} i_{p2} = (E_m I_m / 2) [\cos \phi - \cos 2 (\omega t + \alpha - 120^\circ \pm \phi / 2)] & (b) \\ p_3 &= e_{p3} i_{p3} = (E_m I_m / 2) [\cos \phi - \cos 2 (\omega t + \alpha + 120^\circ \pm \phi / 2)] & (c) \end{aligned} \right\} \quad (59)$$

Now the three double-frequency cosine functions in Eqs. (59a), (59b), and (59c) have equal amplitudes ($E_m I_m / 2$) and are displaced from each other by 120° . Hence their sum is zero. The total instantaneous power supplied by a three-phase balanced circuit is:

$$p = p_1 + p_2 + p_3 = 3 (E_m I_m / 2) \cos \phi = 3 E_p I_p \cos \phi \quad (60)$$

It is significant that although the instantaneous power per phase is pulsating (consists of a constant term, on which is superimposed a double-frequency cosine term) the total instantaneous power in the three-phase balanced circuit is constant and may be represented by Eq. (60). Attempts have been made to obtain polyphase power by using a single-phase source. The reader can foresee the futility of such attempts. Indeed single-phase power is, by its very nature, a pulsating power whereas polyphase power (in balanced p -phase circuits, ($p = 3$) is steady. No contraption can transform one into the other.

(b) *Average Power.* — The average power consumed by each phase of the load in a three-phase balanced circuit may be computed either by using the expression $P_p = R_p I_p^2$ or by using the relation $P_p = E_p I_p \cos \phi_p = \mathbf{E} \bar{\mathbf{I}}$ (real). In the first expression R_p is the resistance per phase and I_p is the effective value of the current per phase flowing through that resistance. In the second expression, E_p is the voltage measured across the load through which the current I_p flows. ϕ is the phase angle between E_p and I_p . We stress this fact because the student is apt to multiply the line-voltage by the line-current and call this product the power per phase. The power per phase is the Emf. per phase multiplied by the current per phase. Remembering this fact we have:

$$P_p = R_p I_p^2 = E_p I_p \cos \phi = \mathbf{E}_p \bar{\mathbf{I}}_p \text{ (real)} \quad (61)$$

Having computed the power consumed per phase the total power consumed by the balanced circuit is:

$$P = 3 P_p = 3 E_p I_p \cos \phi_p \quad (62)$$

(c) *Apparent Power.* — The apparent power per phase is:

$$P_a = E_p I_p \quad (63)$$

and the total apparent power supplied is:

$$P_T = 3 P_a = 3 E_p I_p \quad (64)$$

It can be shown that whenever the phases are interconnected, as in systems of types B, C, D, and E (Figs. 11a to 14a inclusive), Eq. (64) reduces to:

$$P_T = 3 E_p I_p = E_L I_L \sqrt{3} \quad (65)$$

We shall prove this relation for systems belonging to type *B*, leaving the other similar proofs as exercises to the reader.

By Figs. (11a) and (11b) the phase- and line-currents are equal; or: $I_p = I_L$. Moreover from Eq. (8b) we have:

$$E_p = E_L / \sqrt{3}$$

Substituting these values in Eq. (64) we obtain Eq. (65).

17. Power in Three-Phase Balanced Circuits with Non-Sine Emfs. —
(a) *Instantaneous Power.* — Let the Emfs. across the three loads of a balanced three-phase circuit, of any type, be:

$$\left. \begin{aligned} e_{p1} &= \sum E_{mn} \sin n(\omega_1 t + \alpha_n) & (a) \\ e_{p2} &= \sum E_{mn} \sin n(\omega_1 t + \alpha_n - 120^\circ) & (b) \\ e_{p3} &= \sum E_{mn} \sin n(\omega_1 t + \alpha_n + 120^\circ) & (c) \end{aligned} \right\} \quad (66)$$

and let the load currents produced by these Emfs. be:

$$\left. \begin{aligned} i_{p1} &= \sum I_{mn} \sin n(\omega_1 t + \alpha_n \pm \phi_n/n) & (a) \\ i_{p2} &= \sum I_{mn} \sin n(\omega_1 t + \alpha_n - 120^\circ \pm \phi_n/n) & (b) \\ i_{p3} &= \sum I_{mn} \sin n(\omega_1 t + \alpha_n + 120^\circ \pm \phi_n/n) & (c) \end{aligned} \right\} \quad (67)$$

The instantaneous power consumed by each phase of the load is:

$$\left. \begin{aligned} p_1 &= \sum E_n I_n [\cos \phi_n - \cos 2n(\omega_1 t + \alpha_n \pm \phi_n/2n)] & (a) \\ p_2 &= \sum E_n I_n [\cos \phi_n - \cos 2n(\omega_1 t + \alpha_n - 120^\circ \pm \phi_n/2n)] & (b) \\ p_3 &= \sum E_n I_n [\cos \phi_n - \cos 2n(\omega_1 t + \alpha_n + 120^\circ \pm \phi_n/2n)] & (c) \end{aligned} \right\} \quad (68)$$

The total power supplied may be found by adding p_1 , p_2 and p_3 in Eqs. (68). Their sum assumes two forms depending on whether n is triplen or non-triplen. For any non-triplen harmonic the cosine functions in Eqs. (68) may be represented by three vectors of equal amplitude and equally displaced from each other by 120° . Their sum is therefore zero and we have:

$$p_v = p_{1v} + p_{2v} + p_{3v} = 3 \sum E_v I_v \cos \phi_v \quad (69)$$

where v is the order of any non-triplen odd harmonic (1st, 5th, 7th, 11th, etc.).

If n is triplen then the three cosine functions in Eq. (68) are in phase and the result of adding the power terms due to triplen harmonics is:

$$p_u = p_{1u} + p_{2u} + p_{3u} = 3 \sum E_u I_u [\cos \phi_u - \cos 2u(\omega_1 t + \alpha_u \pm \phi_u/2u)] \quad (70)$$

Thus although instantaneous polyphase power due to non-triplen harmonics is steady (see Eq. 69) that due to triplen harmonics (Eq. 70)

is pulsating and resembles single-phase power in this respect. Indeed this is exactly what we would expect for we have already seen that triplen harmonics in the three phases are not 120° out of phase but are actually in phase with each other.

(b) *Average Power.* — Adding Eqs. (68) and determining the average power in the usual manner (see Eq. (6), Chapter XII) we have:

$$P = \sum E_{np} I_{np} \cos \phi_{np} = \sum R I_{np}^2 \quad (71)$$

It must be here emphasized that the triplen harmonics produce power only in circuits belonging to type *A* (see Fig. 12*a*), and in circuits belonging to type *B* when the neutrals *O* and *N* (Fig. 15*a*) are connected. As there exist no triplen harmonic currents in circuits of types *C*, *D*, and *E*; and in circuits of type *B* when the neutrals are isolated, no triplen harmonic power is supplied to the loads of these circuits.

(c) *Apparent Power.* — The apparent power per phase, when the Emfs. of the balanced circuit comprise harmonics, is:

$$P_{ap} = E_p I_p = \left(\sum E_{np}^2 \right)^{1/2} \left(\sum I_{np}^2 \right)^{1/2} \quad (72)$$

where E_{np} and I_{np} are the phase-voltage and phase-current respectively of the n th harmonic.

Here again the triplen harmonics must not be included in the summation (Eq. 48) for circuits belonging to type *B* (neutrals isolated), and to types *C*, *D* and *E*.

18. Conclusion. — The balanced three-phase circuit has been treated in the text in order to familiarize the student with the general method of treating balanced polyphase circuits. Indeed the treatment of the general p -phase balanced circuit would (in so far as the current-voltage relations are concerned) seem superfluous on account of our cyclic choice of notations. Hence the extension of this treatment to the balanced p -phase circuit is given in the form of problems.

PROBLEMS

1. Sketch figures similar to Fig. 1 showing a four- and a six-phase machine.
- 2a. Give the instantaneous value expressions for the balanced phase-voltages of a four-phase and a six-phase alternator assuming that the instantaneous value of the Emf. of the first phase is $e_{1,1} = E_m \sin(\omega t + \alpha)$. The phase sequence is 1, 2, 3, 4.*
- 2b. Write the expressions for the vectors representing the sine Emfs. of problem 2a.

* By the phrase "The phase sequence is 1, 2, 3, 4" we mean that phase 1 leads phase 2, phase 2 leads phase 3, and phase 3 leads phase 4. In other words, the phases lead each other in the order of their numbering.

3. Some rotary converters are built for twelve phases. What would be the angle between any two successive phases, if the machine is balanced. Give the instantaneous and vector expressions for the sine Emf. of each phase assuming that the Emf. of the first phase is $e_{1/1} = E_m \sin (\omega t + \alpha)$, and that the phase sequence is 1, 2, 3 12.

4. Give a schematic diagram similar to Fig. 6a showing a six-phase source connected in star.

5a. Write the vector expressions for the phase- and line-voltages of a balanced six-phase star-connected source, assuming the sequence of phases to be 1, 2, 3 . . . 6.

5b. Draw a vector diagram showing the phase- and line-voltages of problem 5a and deduce from the vector diagram the relation $E_L = E_p$.

6a. Draw a schematic diagram showing a six-phase mesh-connected source. Assume the sequence of the phases to be 1, 2, 3 . . . 6.

6b. Referring to problem 6a, express the vector line-voltages $\mathbf{E}_{12}$, $\mathbf{E}_{23}$. . . $\mathbf{E}_{61}$ in terms of the phase-voltages $\mathbf{E}_{1/1}$, $\mathbf{E}_{2/2}$. . . $\mathbf{E}_{6/6}$.

6c. Referring to Fig. 8, suppose that the three-phase source is so connected in mesh that the terminals 1' and 2; 2' and 3; 3' and 1 are respectively connected together.

(a) Give the line-voltages $\mathbf{E}_{12}$, $\mathbf{E}_{23}$ and $\mathbf{E}_{31}$ in terms of the phase-voltages $\mathbf{E}_{1/1}$, $\mathbf{E}_{2/2}$ and $\mathbf{E}_{3/3}$.

(b) Assuming that the Emf. $\mathbf{E}_{1/1} = E \epsilon^{j\alpha}$, give the complex expressions for the line-voltages.

(c) Show that the vectors representing the line Emfs. form a closed triangle.

7. Draw diagrams of connection of a four-phase circuit belonging to (1) type A, (2) type B, (3) type C, (4) type D, and (5) type E.

8a. The phase-voltages of a three-phase, type A, balanced circuit (Fig. 10a) have effective values equal to 110 volts. The load impedance is $Z = 9 + j8$. The Emf. of the first phase is $\mathbf{E}_{1/1} = 110 \angle 60^\circ$, and the sequence of phases is 1, 2, 3.

(a) Give the complex expressions for the Emfs. of phases 2 and 3.

(b) Compute the current supplied by each phase.

(c) Draw a vector diagram showing each of the Emf. and current vectors.

8b. The phase-voltages of a four-phase, type A, balanced circuit have effective values equal to 110 volts. Each of the four impedances of the load is $Z = 10 + j5$. Assuming that the vector Emf. of phase 1 is $\mathbf{E}_{1/1} = 110 + j0$:

(a) Give the complex expressions for the remaining three Emf. vectors, assuming a phase sequence of 1, 2, 3, 4.

(b) Compute the currents flowing through each phase.

(c) Draw a vector diagram showing the voltage and current vectors.

9a. A balanced three-phase star source supplies power to a balanced, three-phase, Y-connected induction motor. Oscillograms show that $\mathbf{I}_{01}$ lags $\mathbf{E}_{01}$ by 25° . Moreover the effective value of the phase-voltages and currents are $E = 110$ volts and $I = 15$ amp.

(a) Determine the equivalent impedance per phase of the motor.

(b) Draw a vector diagram showing the line-voltages, the line-currents and the phase angle ϕ , assuming that $\mathbf{E}_{01} = 110 \angle 50^\circ$.

9b. A four-phase, balanced, star-star circuit (type *B*) has impedances $Z = 15 + j8$. The Emf. of the first phase is $E_{01} = 220 \angle 45^\circ$.

- Give the complex expressions for the Emfs. of the remaining three phases.
- Compute the line- (phase-) currents.
- Draw a complete vector diagram.

9c. A six-phase, balanced, star-star circuit (type *B*) has impedances $Z = 12 - j7$. The Emf. of the second phase is $E_{02} = 110 \angle 30^\circ$.

- Give the complex expressions for the Emfs. of the remaining five phases, assuming a phase sequence of 1, 2, 3, 4, 5, 6.
- Compute the line- (phase-) currents.
- Draw a complete vector diagram.

10a. A three-phase, balanced, mesh-star circuit (type *C*) has impedances $Z = 10 + j6$. The Emf. between lines 1 and 2 is $E_{12} = 110 \angle 25^\circ$.

- Give the complex expression for the Emfs. E_{23} and E_{31} , assuming that E_{23} lags E_{12} by 120° and that E_{31} leads E_{12} by 120° .
- Give the values of the Emfs. across each of the load-impedances Z .
- Compute the line-currents.
- Draw a complete vector diagram showing line-voltages, line-currents, and the Emfs. across each of the branches of the load.

10b. Develop expressions similar to Eqs. (22) to (25) for a balanced, four-phase, mesh-star circuit.

10c. Develop expressions similar to Eqs. (22) to (25) for a balanced, six-phase, mesh-star circuit.

10d. A balanced, four-phase, mesh-star circuit (type *C*) has impedances $Z = 8 - j5$. The Emf. between lines 1 and 2 is $E_{12} = 110 \angle 30^\circ$.

- Give the complex expressions for the Emfs. E_{23} , E_{34} and E_{41} assuming that these Emfs. lead each other in the order in which they are mentioned.
- Find the values of the Emfs. across the load-impedances.
- Compute the line-currents.
- Draw a complete vector diagram showing line-voltages, line-currents and the Emfs. across each of the branches of the load.

10e. A six-phase, balanced, mesh-star circuit (type *C*) has impedances $Z = 6 + j4$. The Emf. between lines 1 and 2 is $E_{12} = 220 \angle 45^\circ$.

- Give the complex expressions for the Emfs. E_{23} , E_{34} . . . E_{61} assuming that these Emfs. lead each other in the order mentioned.
- Determine the values of the Emfs. across the load-impedances.
- Compute the line-currents.
- Draw a complete vector diagram.

11a. The line voltage E_{12} in a balanced, three-phase, mesh-mesh circuit is $E_{12} = 220 \angle -70^\circ$. The load consists of three delta-connected molten-metal troughs having a resistance $R = 0.2$ ohm per trough.

- Compute the phase- and line-currents of the circuit.
- Draw a complete vector diagram.

11b. Develop expressions similar to Eqs. (26) and (27) for:

- (a) A balanced, four-phase, mesh-mesh circuit.
- (b) A balanced, six-phase, mesh-mesh circuit.

11c. Show that in a balanced, four-phase, mesh-mesh circuit the effective values of the line-currents I_L and of the phase-currents I_p are such that $I_L = \sqrt{2} I_p$.

11d. Show that in a balanced, six-phase, mesh-mesh circuit the effective values of the line-currents I_L and of the phase-currents I_p are such that $I_L = I_p$.

11e. In a balanced, four-phase, mesh-mesh circuit the branch of the load between points 1 and 2 is disconnected. Give the effective values of the line-currents (after Z_{12} is disconnected) in terms of the effective values of the line-currents when Z_{12} was in the circuit.

12a. The Emf. of the first phase in a balanced, three-phase, star-mesh circuit (type E) is $E_{01} = 127 \angle 30^\circ$. Each branch of the delta-connected load has an impedance $Z = 7 \angle 25^\circ$. The sequence of phases is 1, 2, 3.

- (a) Give the vector expressions for the line-voltages.
- (b) Compute the load-currents.
- (c) Determine the line-currents.

12b. Develop expressions similar to Eqs. (29) and (30) for:

- (a) A balanced, four-phase, type E circuit.
- (b) A balanced, six-phase, type E circuit.

13a. Convert the circuits specified in problems 10a, 11a, and 12a to equivalent star-star circuits; then compute the line-currents, and check the results so obtained with the line-currents determined before the conversion.

13b. Assume that the phase- (line-) voltage E_{12} of a balanced, four-phase, mesh source is $E_{12} = E_L \angle \alpha$. Show that the phase-voltage E_{01} of an equivalent balanced four-phase star source is $E_{01} = (E_L/\sqrt{2}) \angle (\alpha + 135^\circ)$.

13c. Show that if Z_m is the impedance of any one branch of a four-phase, mesh-connected load, then the impedance, per branch, of the equivalent four-phase, star-connected load is $Z_s = 0.5 Z_m$.

13d. A balanced 220-volt, four-phase, mesh-mesh circuit has load impedances $Z_m = 12 \angle 30^\circ$. Assuming that the line-voltage $E_{12} = 220 \angle 45^\circ$; and that the phase sequence is 1, 2, 3, 4:

- (a) Convert this circuit into an equivalent balanced star-star circuit.
- (b) Compute the line-currents.

13e. Let the phase- (line-) voltage E_{12} of a six-phase balanced, mesh source be: $E_{12} = E_L \angle \alpha$. Show that the phase-voltage E_{01} of an equivalent balanced, six-phase star source is $E_{01} = E_L \angle (\alpha + 120^\circ)$.

13f. Show that if Z_m is the impedance of any one branch of a six-phase mesh-connected load, then the impedance, per branch, of the equivalent six-phase star-connected load is $Z_s = Z_m$.

Solution. — Draw a sketch similar to Fig. 14a for a six-phase circuit. Let the mesh load-currents be $I_{12} = E_{12}Z_m \dots I_{16} = E_{16}Z_m$ and let the line-currents be $I_{01}, I_{02} \dots I_{06}$. Applying Kirchhoff's current law to the point 1 we have:

$$I_{01} = I_{61} - I_{12} = (E_{61} - E_{12})/Z_m \quad (a)$$

Now draw another sketch similar to Fig. 6b for a six-phase source. Observe that $\mathbf{E}_{01} = \mathbf{E}_{01} - \mathbf{E}_{06}$ and $\mathbf{E}_{12} = \mathbf{E}_{02} - \mathbf{E}_{01}$. Making these substitutions in Eq. (a) we have:

$$\mathbf{I}_{01} = [2 \mathbf{E}_{01} - (\mathbf{E}_{06} + \mathbf{E}_{02})]/Z_m \quad (b)$$

But $\mathbf{E}_{06} + \mathbf{E}_{02} = \mathbf{E}_{01}$ hence Eq. (b) becomes:

$$\mathbf{I}_{01} = \mathbf{E}_{01}/Z_m \quad (c)$$

Now in a star-star system:

$$\mathbf{I}_{01} = \mathbf{E}_{01}/Z_s \quad (d)$$

If the two systems are to be equivalent then $\mathbf{I}_{01}$ (in Eq. c) must be identical with $\mathbf{I}_{01}$ (in Eq. d), which means that:

$$Z_s = Z_m \quad \text{Q.E.D.} \quad (e)$$

13g. In a balanced, six-phase (type *D*) circuit the line-voltage $\mathbf{E}_{12} = 220 \angle 30^\circ$. The impedance per branch of the mesh-connected load is $Z_m = 15 \angle 25^\circ$.

(a) Give the complex expressions for the phase-voltages of the equivalent balanced star source.

(b) Compute the line-currents.

13h. The line impedances, throughout this work, have been assumed negligible. Although this may be the condition in a laboratory experiment, it is not true in practice. Deduce the Emf.-current relations in a balanced three-phase circuit (type *B*) whose line-impedances are Z_L and whose load-impedances are Z_s . Show that for this type of circuit the line- and the load-impedances are in series and may thus be regarded as one impedance $Z = Z_L + Z_s$.

13i. The advantage of converting polyphase circuits of types *C*, *D*, or *E* into an equivalent star-star circuit becomes obvious when the line-impedances are not negligible. In order to illustrate this assume in problem 12a that the impedance per line is $2 \angle 30^\circ$.

(a) Compute the line-currents by the use of Kirchhoff's laws.

(b) Change the circuit into an equivalent star-star circuit and compute the line currents.

(c) Compare the two methods and decide which is the simpler.

14a. Show that the third harmonic Emfs., in a four-phase balanced source, have a phase sequence of 1, 4, 3, 2, and that the Emfs. in the four phases are displaced from each other by 90° .

14b. Let the order u of any triplen harmonic be expressed by the equation $u = (6n + 3)$. Show in general that in a four-phase balanced circuit if n is zero or an even integer then the phase sequence of the triplen harmonic is 1, 4, 3, 2. If, however, n is an odd integer then the phase sequence is 1, 2, 3, 4.

14c. Show that the phase sequence of the fifth harmonic Emf. in a four-phase balanced source is 1, 2, 3, 4.

14d. Verify the following rule regarding the phase sequence of any non-triplen harmonic in a four-phase balanced circuit.

RULE: Let the order v of a non-triplen harmonic in a balanced four-phase circuit be expressed by the equation $v = (6n \pm 1)$. Then:

(a) The sequence is 1, 2, 3, 4 if (1) $v = 6n + 1$ $n = 0$ or an even integer
(2) $v = 6n - 1$ $n = \text{an odd integer}$

(b) The sequence is 1, 4, 3, 2 if (1) $v = 6n + 1$ $n = \text{an odd integer}$
(2) $v = 6n - 1$ $n = \text{an even integer}$

14e. Show that in a six-phase balanced circuit the triplen harmonics in alternate phases are in phase with each other and that these harmonics are displaced from each other by 180° in successive phases.

14f. Show that in a six-phase balanced circuit the non-triplen harmonics whose order v is expressed by the equation $v = 6n + 1$ have the same phase sequence as the fundamental; and that the non-triplen harmonics whose order is expressed by $v' = 6n - 1$ have a negative sequence, that is, the second phase leads (rather than lags) the first by 60° ; the third phase leads the second by 60° , etc.

15a. The Emfs. of a balanced three-phase type A circuit have a 10 per cent fifth harmonic. Each branch of the load has an impedance Z_1 which, at fundamental frequency, is $Z_1 = 8 + j3$. Assuming that the instantaneous value of the Emf. of the first phase is:

$$e_{01} = (110 \sqrt{2}) \sin (\omega_1 t + 30^\circ) + (5.5 \sqrt{2}) \sin 5 (\omega_1 t + 6^\circ)$$

Let the phase sequence of the fundamental be 1, 2, 3.

- Give the instantaneous values of the phase Emfs. e_{02} and e_{03} .
- Find the line currents due to the fundamental.
- Compute the line-currents due to the fifth harmonic.
- Give the instantaneous values of the line-currents i_{01} , i_{02} , and i_{03} .
- Determine the effective value of any phase Emf.
- Determine the effective value of any line-current.

Solution. —

- (a) The instantaneous values of the phase Emfs. are:

$$\begin{aligned} e_{01} &= 156 \sin (\omega_1 t + 30^\circ) + 7.8 \sin 5 (\omega_1 t + 6^\circ) \\ e_{02} &= 156 \sin (\omega_1 t - 90^\circ) + 7.8 \sin 5 (\omega_1 t - 114^\circ) \\ e_{03} &= 156 \sin (\omega_1 t + 150^\circ) + 7.8 \sin 5 (\omega_1 t + 126^\circ) \end{aligned}$$

- (b) Line-currents due to the fundamental:

$$\begin{aligned} \mathbf{I}_{01} &= \mathbf{E}_{01}/Z_1 = 110 \angle 30^\circ / 8.55 \angle 20.5^\circ = 12.88 \angle 9.5^\circ \\ \mathbf{I}_{02} &= 12.88 \angle (9.5^\circ - 120^\circ) = 12.88 \angle -110.5^\circ \\ \mathbf{I}_{03} &= 12.88 \angle (9.5^\circ + 120^\circ) = 12.88 \angle 129.5^\circ \end{aligned}$$

- (c) Line-currents due to the fifth harmonic. The fifth harmonic Emfs. in the three phases are:

$$\mathbf{E}_{01} = 110 \times 0.05 \angle 30^\circ = 5.5 \angle 30^\circ, \quad \mathbf{E}_{02} = 5.5 \angle 150^\circ, \quad \mathbf{E}_{03} = 5.5 \angle -90^\circ$$

where all the angles are measured to the scale of the fifth harmonic.

The impedance Z_5 to the fifth harmonic is $8 + j15$. Consequently the fifth harmonic currents are:

$$\begin{aligned} \mathbf{I}_{01} &= 5.5 \angle 30^\circ / 17 \angle 62^\circ = 0.324 \angle -32^\circ \\ \mathbf{I}_{02} &= 0.324 \angle 88^\circ \\ \mathbf{I}_{03} &= 0.324 \angle -152^\circ \end{aligned}$$

where all the angles are measured to the scale of the fifth harmonic.

- (d) The instantaneous values of the line-currents are:

$$\begin{aligned} i_{01} &= (12.88 \sqrt{2}) \sin (\omega_1 t + 9.5^\circ) + (0.324 \sqrt{2}) \sin 5 (\omega_1 t - 64^\circ) \\ i_{02} &= 18.22 \sin (\omega_1 t - 110.5^\circ) + 0.458 \sin 5 (\omega_1 t + 17.6^\circ) \\ i_{03} &= 18.22 \sin (\omega_1 t + 129.5^\circ) + 0.458 \sin 5 (\omega_1 t - 30.4^\circ) \end{aligned}$$

- (e) $E = \sqrt{E_1^2 + E_5^2} = 110.02$

- (f) $I = \sqrt{I_1^2 + I_5^2} = 12.9$

15b. A balanced three-phase star-star circuit has a 10 per cent third and a 6 per cent seventh harmonic. The instantaneous value of the Emf. in the first phase is:

$$e_{01} = E_m \sin (\omega_1 t + 25^\circ) + 0.1 E_m \sin 3 (\omega_1 t + 10^\circ) + 0.06 E_m \sin 7 (\omega_1 t + 5^\circ)$$

The phase sequence is 1, 2, 3.

The effective value of the fundamental is 110 volts and the impedance (to the fundamental) per branch of the balanced star load is $Z_1 = 7 + j2$.

- Give the instantaneous values of the Emfs. in the second and third phases.
- Determine the effective values of the line-currents due to the fundamental and seventh harmonic.
- Give the instantaneous values of the three line-currents.
- Compute the effective value of the Emf. per phase.
- Compute the effective value of the current per phase.
- Give the effective value of the voltage E_{0N} .

15c. A balanced, three-phase, mesh-star circuit has in its line Emfs. a fundamental and a 6 per cent fifth harmonic. The effective value of the fundamental is 110 volts and the instantaneous value of the line-voltage e_{12} is:

$$e_{12} = (110 \sqrt{2}) \sin (\omega t + 45^\circ) + (6.6 \sqrt{2}) \sin 5 (\omega t + 10^\circ)$$

The impedance (to the fundamental) of each branch is $Z_1 = 10 + j4$. The phase sequence of the fundamental is 1, 3, 2.

- Give the instantaneous values of the Emfs. e_{23} and e_{31} .
- Determine the effective values of the line-currents due to the fundamental and to the fifth harmonic.
- Give the instantaneous values of these line-currents.
- Compute the effective values of the line-voltages.
- Evaluate the effective values of the line-currents.

15d. The line-voltage e_{12} of a balanced, mesh-mesh circuit is:

$$e_{12} = (110 \sqrt{2}) \sin (\omega_1 t + 20^\circ) + (4.4 \sqrt{2}) \sin 11 (\omega_1 t + 5^\circ)$$

The impedance to the fundamental of each branch of the mesh-connected load is $Z_1 = 15 + j5$. The phase sequence of the fundamental is 1, 3, 2.

- Give the instantaneous values of the Emfs. e_{23} and e_{31} .
- Determine the effective values of the phase-currents due to the fundamental and to the eleventh harmonic.
- Determine the effective values of the line-currents.
- Give the instantaneous values of the line- and phase-currents.
- Find the effective values of the line-voltages.
- Find the effective values of the line- and phase-currents.

15e. The Emf. of the first phase in a four-phase, star-star circuit is:

$$e_{01} = (110 \sqrt{2}) \sin (\omega_1 t + 20^\circ) + (8.8 \sqrt{2}) \sin 3 (\omega_1 t + 10^\circ) + (4.4 \sqrt{2}) \sin 5 (\omega_1 t + 5^\circ)$$

The impedance (to the fundamental) of each branch of the star connected load is $Z_1 = 12 + j4$. The phase sequence of the fundamental is 1, 2, 3, 4.

- Give the instantaneous values of the phase Emfs. e_{02} , e_{03} and e_{04} .
- Compute the effective values of the line-currents due to each of the three harmonics.

- (c) Give the instantaneous values of the current in each line.
- (d) Determine the effective value of the current in any line.
- (e) Evaluate the effective value of the Emf. in any phase.

16a. Show that Eqs. (64) and (65) are identical for any type of interconnected balanced three-phase circuit.

16b. Determine the instantaneous, average, and apparent power for the circuits specified in problems 8a, 8b, 9a, 9b, 9c, 10a, 10d, 10e, 11a and 12a.

17. Determine the instantaneous, average, and apparent power for the circuits given in problems 15a to 15e.

CHAPTER XVI

POLYPHASE CIRCUITS

PART B. UNBALANCED CIRCUITS

1. Polyphase Transformations. — A polyphase Emf. source is said to be transformed when the individual phase Emfs. are so interconnected that the number of phases in the interconnected sources is not the same as the number of individual Emfs. A few typical transformations will be described in order to illustrate this definition.

(a) *Two-Phase to Four-Phase Transformation.* — Consider a two-phase Emf. source (Fig. 1a) in which phase 1 leads phase 2 by 90° . Let the terminals of the first phase be 1'1 and those of the second phase be 2'2. Let taps, 0, be provided in the middle of each phase. The Emf. of each phase may now be regarded as consisting of two parts (Figs. 1a and 1b) such that:

$$\left. \begin{aligned} \mathbf{E}_{1'1} &= \mathbf{E}_{01} - \mathbf{E}_{01'} & (a) \\ \mathbf{E}_{2'2} &= \mathbf{E}_{02} - \mathbf{E}_{02'} & (b) \end{aligned} \right\} \quad (1)$$

Now let the taps 0 be joined together (Fig. 1c) and the terminals 1, 1', 2, and 2' be brought out. This gives a four-phase star-connected source. The phase- and line-voltages of this source are shown in Fig. 1d. These are:

$$\left. \begin{aligned} \mathbf{E}_{01} &= 0.5 \mathbf{E}_{1'1} & (a) & \quad \mathbf{E}_{12} = \mathbf{E}_{02} - \mathbf{E}_{01} & (e) \\ \mathbf{E}_{02} &= 0.5 \mathbf{E}_{2'2} & (b) & \quad \mathbf{E}_{23} = \mathbf{E}_{03} - \mathbf{E}_{02} & (f) \\ \mathbf{E}_{03} &= -0.5 \mathbf{E}_{1'1} & (c) & \quad \mathbf{E}_{34} = \mathbf{E}_{04} - \mathbf{E}_{03} & (g) \\ \mathbf{E}_{04} &= -0.5 \mathbf{E}_{2'2} & (d) & \quad \mathbf{E}_{41} = \mathbf{E}_{01} - \mathbf{E}_{04} & (h) \end{aligned} \right\} \quad (2)$$

(b) *Two-Phase to Three-Phase Transformation (The Scott or T-Connection).* — Fig. 2a shows a two-phase source in which phase 1 leads phase 2 by 90° . Let the effective value of the Emf. in phase 1 be E and that in phase 2 be $(\sqrt{3}/2) E$. Let a tap 0 be provided in the middle of the first phase. The first phase may be regarded as consisting of $\mathbf{E}_{01}$ and $\mathbf{E}_{02}$ (Fig. 2b)* such that

$$\mathbf{E}_{1'1} = \mathbf{E}_{01} - \mathbf{E}_{01'} \quad (3)$$

Now let the points 0 and 2' (Fig. 2a) be connected together and let the terminals 1, 2 and 1' be brought out as shown in Fig. 2c. This

* The similarity of this figure to the letter T is responsible for the name of this type of connection.

gives a three-phase star-connected source. The phase- and line-voltages of this source are shown in Fig. 2d. These are:

$$\left. \begin{aligned} \mathbf{E}_{01} &= 0.5 \mathbf{E}_{1'1} & (a) \\ \mathbf{E}_{02} &= \mathbf{E}_{2'2} & (b) \\ \mathbf{E}_{03} &= -0.5 \mathbf{E}_{1'1} & (c) \end{aligned} \right\} \quad \left. \begin{aligned} \mathbf{E}_{12} &= \mathbf{E}_{02} - \mathbf{E}_{01} & (d) \\ \mathbf{E}_{23} &= \mathbf{E}_{03} - \mathbf{E}_{02} & (e) \\ \mathbf{E}_{31} &= \mathbf{E}_{01} - \mathbf{E}_{03} & (f) \end{aligned} \right\} \quad (4)$$

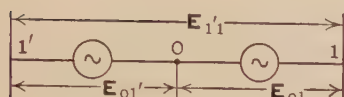


Fig. 1a

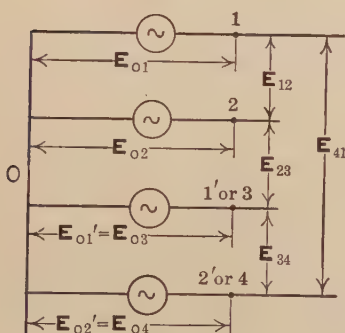


Fig. 1c

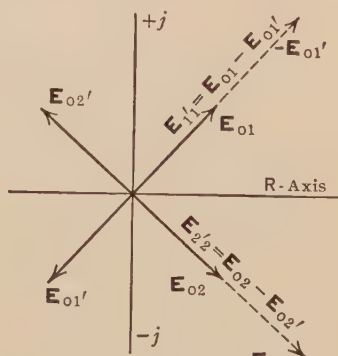


Fig. 1b

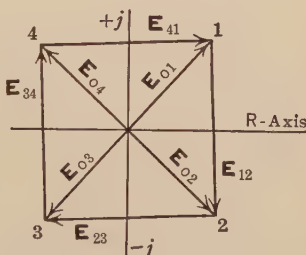


Fig. 1d

FIG. 1a. A two-phase Emf. source with taps 0 in the middle of each phase.

FIG. 1b. Vector diagram of the Emfs. in Fig. 1a.

FIG. 1c. Transformation of the two-phase Emf. source, Fig. 1a, into a quarter-phase or four-phase, star-connected, balanced source.

FIG. 1d. Vector diagram of the phase and line Emfs. in Fig. 1c.

Although the phase-voltages ($\mathbf{E}_{01}$, $\mathbf{E}_{02}$ and $\mathbf{E}_{03}$) of this type of connection are neither equal nor equally displaced from each other, yet the line-voltages $\mathbf{E}_{12}$, $\mathbf{E}_{23}$ and $\mathbf{E}_{31}$ are balanced. That such is the case may be easily shown by the following simple considerations.

By assumption:

$$\left. \begin{aligned} E_{1'1} &= E & (a) \\ E_{01} &= E_{03} = 0.5 E & (b) \\ E_{02} &= (\sqrt{3}/2) E & (c) \end{aligned} \right\} \quad (5)$$

Hence by the geometry of Fig. 2d:

$$E_{12} = E_{23} = \sqrt{(0.5 E)^2 + (\sqrt{3}/2)^2 E^2} = E = E_{1'1} \quad (6)$$

Eq. (6) shows that the effective values of the line-voltages are equal. To show that these voltages are equally displaced from each other by

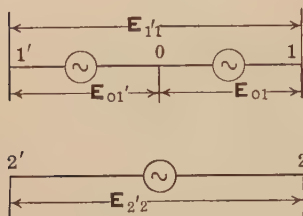


Fig. 2a

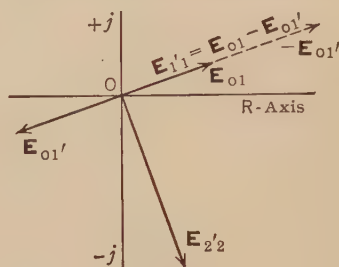


Fig. 2b

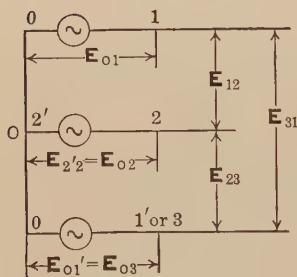


Fig. 2c

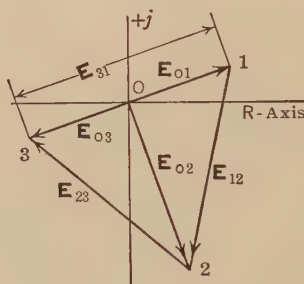


Fig. 2d

FIG. 2a. A two-phase Emf. source with a tap 0 in the middle of phase 1.

FIG. 2b. Vector diagram of the Emfs. in Fig. 2a.

FIG. 2c. Transformation of the two-phase Emf. source Fig. 2a into a three-phase star-connected source.

FIG. 2d. Vector diagram of the phase and line Emfs. in Fig. 2c.

120° we shall prove that their sum is zero. Thus adding Eqs. (4d), (4e) and (4f) we have:

$$E_{12} + E_{23} + E_{31} = 0 \quad (7)$$

(c) *Two-Phase to Three-Phase Transformation (The V-Connection.*)*—Consider a two-phase source (Fig. 3) in which phase 1 leads phase 2 by 60° . Let the terminals $1'$ and $2'$ be joined together to a common point 0 (Fig. 3c) and let the terminals 1, 2, and 0 (or 3) be brought out as three terminals. The vector diagram representing the phase- and

* The similarity of the vector diagram (Fig. 3b) to the letter V is responsible for the name given to this type of connection.

line-voltages of this type of connection is shown in Fig. 3d. Since 0, 1', 2' and 3 are one and the same point (see Fig. 3c) we have the following expressions for the phase-voltages:

$$\left. \begin{aligned} \mathbf{E}_{01} &= \mathbf{E}_{1'1} & (a) \\ \mathbf{E}_{02} &= \mathbf{E}_{2'2} & (b) \\ \mathbf{E}_{03} &= 0 & (c) \end{aligned} \right\} \quad (8)$$

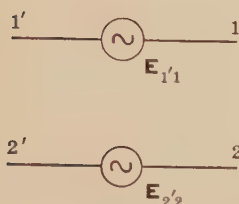


Fig. 3a

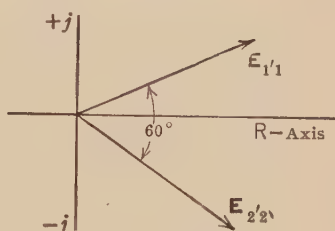


Fig. 3b

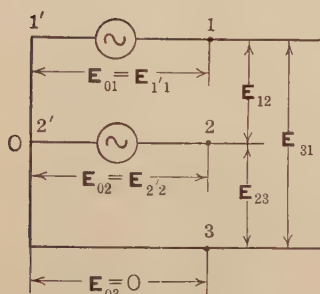


Fig. 3c

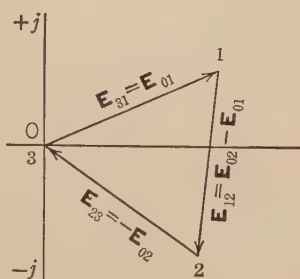


Fig. 3d

FIG. 3a. A two-phase source in which the Emfs. are equal and displaced from each other by 60° .

FIG. 3b. Vector diagram of the Emfs. in Fig. 3a.

FIG. 3c. Transformation of the two-phase Emf. source Fig. 3a into a three-phase star-connected source.

FIG. 3d. Vector diagram of the phase and line Emfs. in Fig. 3c.

The line-voltages may be expressed in terms of the phase-voltages by referring to the vector diagram Fig. 3d. Thus:

$$\left. \begin{aligned} \mathbf{E}_{12} &= \mathbf{E}_{02} - \mathbf{E}_{01} & (a) \\ \mathbf{E}_{23} &= -\mathbf{E}_{02} & (b) \\ \mathbf{E}_{31} &= \mathbf{E}_{01} & (c) \end{aligned} \right\} \quad (9)$$

Here again the line-voltages are balanced (that is, they are equal and equally displaced from each other by 120°). This is not true of the phase-voltages.

We shall not pursue this subject any further because the cases are innumerable. We have shown two cases (the T- and the V-connection which lead to unbalanced polyphase sources that are very commonly used in practice. It is needless to assert that unbalanced polyphase loads are the rule, rather than the exception, in commercial practice. Hence a general treatment of unbalanced polyphase circuits is inevitable.*

2. Types of Polyphase Circuits. — Polyphase circuits will, for the purpose of treatment, be divided (as in Chapter XIV) into five types as shown in Table I.

TABLE I. TYPES OF POLYPHASE CIRCUITS

Type of circuit	Methods of connection		Name of circuit	Fig.
	Source	Load		
<i>A</i>	no inter-connection	no inter-connection	non-inter-connected	
<i>B</i>	star	star	star-star	4
<i>C</i>	mesh	star	mesh-star	5
<i>D</i>	mesh	mesh	mesh-mesh	6
<i>E</i>	star	mesh	star-mesh	7

In order to make this work general we shall treat the unbalanced p -phase circuit with balanced or unbalanced sources and balanced or unbalanced loads. The problem in its simplest terms is as follows:

- Given: (1) The star or mesh (line) vector voltages which may or may not be balanced.
 (2) The line- and load-impedances (balanced or unbalanced).

- Required: (1) The line-currents.
 (2) The load-currents.

3. Current and Voltage Relations in a Polyphase Circuit Belonging to Type A (Non-Interconnected). — When the sources and load are not electrically interconnected each phase may be solved as an independent

* The treatment given in this chapter is based solely on the application of Kirchhoff's Emf. and current laws. No new concepts are introduced. Other treatments of this topic using the so-called "Method of Symmetrical Components" may be found in the following named papers:

L. G. Stokvis, Elec. World, 1915, Vol. 65, p. 1111. C. L. Fortescue, A.I.E.E. Trans., 1918, Vol. 37, part 2, p. 1027. V. Genkin, Revue Général de L'Elec., 1924, Vol. 16, pp. 907-951.

circuit. By the application of the relation $\mathbf{E} = \mathbf{I}\mathbf{Z}$ where $\mathbf{E}$ is the vector Emf. of any one phase and $\mathbf{Z}$ the load of that phase.

4. Current and Voltage Relations in a Polyphase Circuit Belonging to Type B (Star-Star). — Fig. 4 shows a star-star m -phase unbalanced circuit. The phase Emfs. $\mathbf{E}_{01}, \mathbf{E}_{02} \dots \mathbf{E}_{0m}$ may or may not be symmetrical. The line-impedances $\mathbf{Z}_{a1}, \mathbf{Z}_{b2} \dots \mathbf{Z}_{mm}$ may or may not be identical and the load-impedances $\mathbf{Z}_{Na}, \mathbf{Z}_{Nb} \dots \mathbf{Z}_{Nm}$ may or may not be balanced. There exists an impedance $\mathbf{Z}_{0N}$ between the source and load neutrals which may assume any value from zero to infinity. Thus if the neutrals 0 and N are grounded $\mathbf{Z}_{0N}$ is practically equal to zero. If 0 and N are isolated (not electrically connected) $\mathbf{Z}_{0N}$ is infinite.

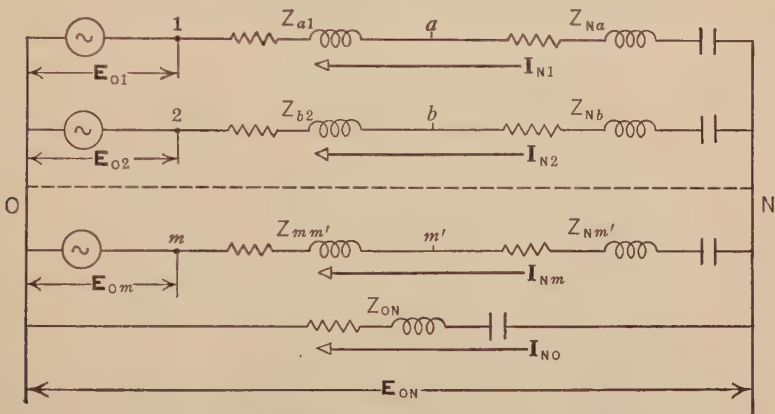


FIG. 4. An m -phase, unbalanced, star-star circuit.

In order to determine the relations between the current and voltage in such a circuit let the voltage between the neutrals 0 and N (Fig. 4) be $\mathbf{E}_{0N}$ and let the combined line- and load-impedances be:

$$\left. \begin{aligned} \mathbf{Z}_1 &= \mathbf{Z}_{a1} + \mathbf{Z}_{Na} & (a) \\ \mathbf{Z}_2 &= \mathbf{Z}_{b2} + \mathbf{Z}_{Nb} & (b) \\ &\vdots & \\ \mathbf{Z}_m &= \mathbf{Z}_{mm'} + \mathbf{Z}_{Nm'} & (m) \end{aligned} \right\} \quad (10)$$

Applying Kirchhoff's Emf. law to the circuits 01aN0, 02bN0 $\dots$ we have:

$$\left. \begin{aligned} \mathbf{E}_{N1} &= \mathbf{E}_{01} - \mathbf{E}_{0N} = \mathbf{I}_{N1}\mathbf{Z}_1 & (a) \\ \mathbf{E}_{N2} &= \mathbf{E}_{02} - \mathbf{E}_{0N} = \mathbf{I}_{N2}\mathbf{Z}_2 & (b) \\ &\vdots & \\ \mathbf{E}_{Nm} &= \mathbf{E}_{0m} - \mathbf{E}_{0N} = \mathbf{I}_{Nm}\mathbf{Z}_m & (m) \\ \mathbf{E}_{N0} &= 0 - \mathbf{E}_{0N} = \mathbf{I}_{N0}\mathbf{Z}_{0N} & (n) \end{aligned} \right\} \quad (11)$$

The attention of the student is called to the fact that the subscripts in successive equations follow a cyclic order. Thus, wherever the subscript 1 occurs in Eq. (a) the subscript 2 takes its place in Eq. (b) etc. This cyclic transformation of subscripts is a great aid in writing polyphase equations and in checking these equations after they have been written. It characterizes all the equations occurring in this and the previous chapter and arises from the symmetry of the choice of notations. Its use is strongly recommended.

Solving Eqs. (11) for the line-currents and remembering that $Y = 1/Z$ we have:

$$\left. \begin{aligned} I_{N1} &= (E_{01} - E_{0N})Y_1 & (a) \\ I_{N2} &= (E_{02} - E_{0N})Y_2 & (b) \\ &\vdots & \\ I_{Nm} &= (E_{0m} - E_{0N})Y_m & (m) \\ I_{N0} &= -E_{0N}Y_{0N} & (N) \end{aligned} \right\} \quad (12)$$

Applying Kirchhoff's current law to the point N (Fig. 4) we have:

$$I_{N1} + I_{N2} + \cdots + I_{Nm} + I_{N0} = 0 \quad (13)$$

Adding Eqs. (12), equating their sum to zero (see Eq. 13) and solving for E_{0N} we have:

$$E_{0N} = \frac{E_{01}Y_1 + E_{02}Y_2 + \cdots + E_{0m}Y_m}{Y_1 + Y_2 + \cdots + Y_m + Y_{0N}} = \frac{\sum E_s Y_s}{Y_{0N} + \sum Y_s} \quad (14)$$

Where E_s is any star- (phase-) voltage and Y_s is the corresponding star admittance these admittances are the reciprocals of the impedances $Z_1, Z_2, \dots, Z_m$ given in Eq. (10).

Eq. (14) gives the value of E_{0N} in terms of the known Emfs. and admittances of the polyphase circuit. The substitution of this value in Eqs. (12) enables one to compute directly the values of the line-currents.

Problem 4a at the end of the chapter is solved in an effort to render this treatment clearer.

5. Current and Voltage Relations in a Polyphase Circuit Belonging to Type C (Mesh-Star). — Fig. 5 shows an m -phase circuit in which a mesh source supplies power to a star load. The combined line- and load-impedances are $Z_1, Z_2, \dots, Z_m$ (see Eqs. 10) and the line-voltages are $E_{12}, E_{23}, \dots, E_{m1}$. Neither the line-voltages nor the impedances are necessarily balanced.

The relations between the currents and voltages in such a circuit may be established as follows:

Applying Kirchhoff's Emf. law to the circuits 1N2, 2N3, etc., we have:

$$\left. \begin{aligned} \mathbf{E}_{12} &= \mathbf{I}_{N2}Z_2 - \mathbf{I}_{N1}Z_1 & (a) \\ \mathbf{E}_{23} &= \mathbf{I}_{N3}Z_3 - \mathbf{I}_{N2}Z_2 & (b) \\ &\vdots & \vdots \\ \mathbf{E}_{Lm} &= \mathbf{I}_{Nm}Z_m - \mathbf{I}_{NL}Z_L & (L) \end{aligned} \right\} \quad (15)$$

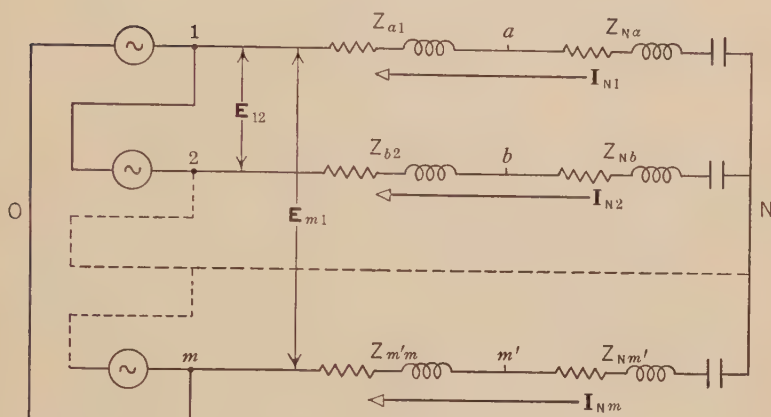


FIG. 5. An m -phase, unbalanced, mesh-star circuit.

Applying Kirchhoff's current law to the point N we have:

$$\mathbf{I}_{N1} + \mathbf{I}_{N2} + \dots + \mathbf{I}_{Nm} = 0 \quad (16)$$

Equations (15) and (16) furnish m equations from which the unknown m -currents may be determined.*

A numerical problem (5a at the end of the chapter) is solved in order to illustrate the use of Eqs. (15) and (16) and to acquaint the student with the solution of simultaneous equations involving complex numbers.

6. Current and Voltage Relations in a Polyphase Circuit of Type D (Mesh-Mesh). — The polyphase circuit shown in Fig. 6 consists of a mesh-connected polyphase source, which supplies power to a mesh-connected load through the line-impedances $Z_{a1}Z_{b2} \dots Z_{m'm}$. The relations between the current and voltage in such a circuit may be established through the use of Kirchhoff's laws. Thus applying Kirchhoff's Emf. law to the circuits 1ab2, 2bc3 . . . we have:

$$\left. \begin{aligned} \mathbf{E}_{12} &= Z_{b2}\mathbf{I}_{b2} + Z_{ab}\mathbf{I}_{ab} - Z_{a1}\mathbf{I}_{a1} & (a) \\ \mathbf{E}_{23} &= Z_{c3}\mathbf{I}_{c3} + Z_{bc}\mathbf{I}_{bc} - Z_{b2}\mathbf{I}_{b2} & (b) \\ &\vdots & \vdots \\ \mathbf{E}_{m1} &= Z_{a1}\mathbf{I}_{a1} + Z_{m'a}\mathbf{I}_{m'a} - Z_{m'm}\mathbf{I}_{m'm} & (m) \end{aligned} \right\} \quad (17)$$

* The use of determinants is recommended in the solution of these equations. Any book on higher algebra would contain a chapter on determinants.

Applying Kirchhoff's current law to the points $a, b, c \dots m'$ we have:

$$\left. \begin{aligned} I_{a1} + I_{ab} - I_{m'a} &= 0 \quad (a) \\ I_{b2} + I_{bc} - I_{ab} &= 0 \quad (b) \\ &\vdots \\ I_{m'm} - I_{m'a} - I_{m'm'} &= 0 \quad (m) \end{aligned} \right\} \quad (18)$$

Equations (17) and (18) give $2m$ vector equations from which the m line-currents and the m load-currents may be determined.

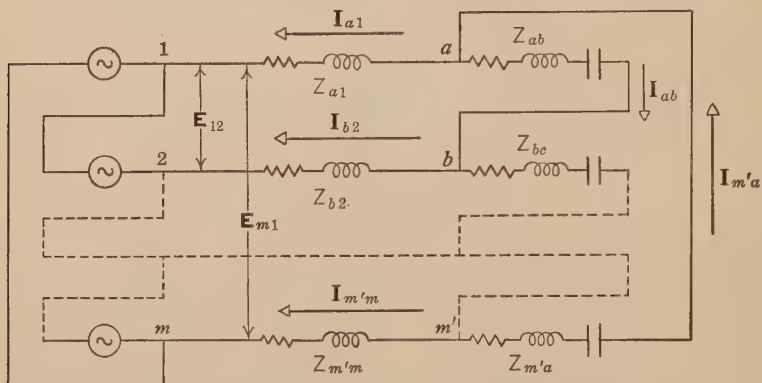


FIG. 6. An m -phase, unbalanced, mesh-mesh circuit.

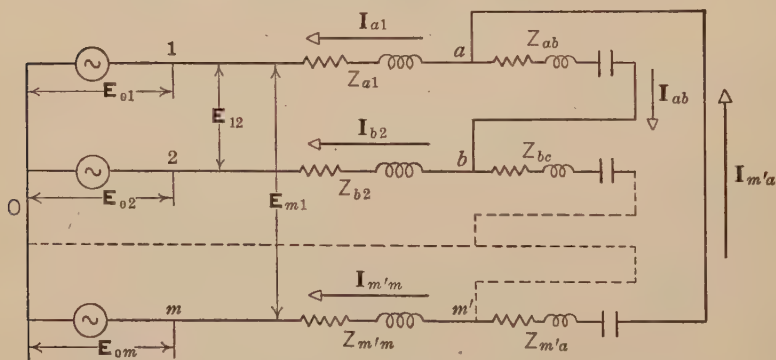


FIG. 7. An m -phase, unbalanced, star-mesh circuit.

7. Current and Voltage Relations in a Polyphase Circuit of Type E (Star-Mesh). — This type of polyphase circuits (Fig. 7) may be reduced to type E by merely determining the line-voltages from the given star voltages. Thus:

$$\left. \begin{aligned} E_{12} &= E_{02} - E_{01} \quad (a) \\ E_{23} &= E_{03} - E_{02} \quad (b) \\ &\vdots \\ E_{m1} &= E_{01} - E_{0m} \quad (m) \end{aligned} \right\} \quad (19)$$

Having determined the line-voltages, the procedure is exactly as outlined in the preceding paragraph because the load connections of this type are identical with the load connections of type *D*.

8. Equivalent Mesh and Star Polyphase Circuits. — The determination of the currents in the last three types of polyphase circuits involves the solution of simultaneous equations. This is, at best, a tedious process. Hence it is often desirable, if possible, to convert a polyphase circuit into an equivalent star-star circuit.

Two polyphase circuits are equivalent when:

- (1) *The vector line-voltages of the two circuits are identical.*
- (2) *The vector line-currents are identical.*

The problem of converting a polyphase circuit of types *C*, *D*, or *E* to an equivalent star-star circuit (type *B*) consists, therefore, in determining one or both of the following:

- (a) A star-connected source whose line-voltages are identical with the given line-voltages of the mesh-connected source.
- (b) A star-connected load which draws the identical vector line-currents as the mesh-connected load or which is equivalent to the mesh load.

Table I shows that:

- (1) Circuits of type *C* have a star load. Hence only the source need be converted to an equivalent star source in order to treat this circuit as a star-star circuit.
- (2) Circuits of type *E* have a star source. Hence if the mesh load is converted to an equivalent star load the circuit becomes a star-star circuit.
- (3) Circuits of type *D* have both the source and the load mesh-connected. Consequently both the source and load should be converted in order to change this circuit to type *B*.

We shall demonstrate how a mesh source and a mesh load may be replaced by an equivalent star source and an equivalent star load respectively.

A. The Equivalent Star Source. — Consider the mesh- (line-) voltages $\mathbf{E}_{12}$, $\mathbf{E}_{23}$. . . $\mathbf{E}_{61}$ given in Figs. 8*a* and 8*b*; and let it be required to determine a set of star voltages $\mathbf{E}_{01}$, $\mathbf{E}_{02}$. . . $\mathbf{E}_{06}$ such that:

$$\left. \begin{array}{l} \mathbf{E}_{02} - \mathbf{E}_{01} = \mathbf{E}_{12} \quad (a) \\ \mathbf{E}_{03} - \mathbf{E}_{02} = \mathbf{E}_{23} \quad (b) \\ \vdots \\ \mathbf{E}_{01} - \mathbf{E}_{06} = \mathbf{E}_{61} \quad (f) \end{array} \right\} \quad (20)$$

Any point 0 (Figs. 8a and 8b) may be taken as the neutral point of the equivalent star voltages. Then by drawing vectors from 0 to the points 1, 2 . . . 6, the vectors $E_{01}, E_{02} \dots E_{06}$ are obtained which satisfy Eq. (20). The selection of point 0 as in Fig. 8a involves an unnecessary amount of computations in order to express the star voltages in terms of the given mesh voltages. Hence the point 0 will be

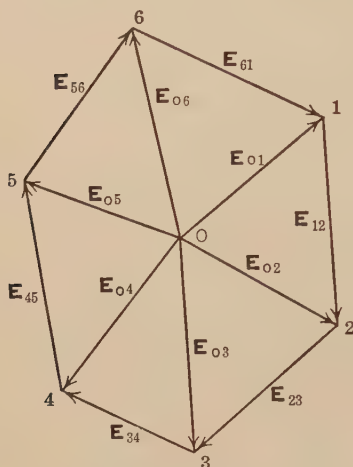


Fig. 8a

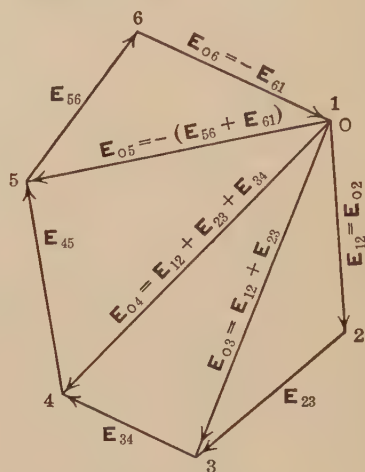


Fig. 8b

FIG. 8a. A set of mesh and of star voltages that are equivalent.

FIG. 8b. The same set of mesh voltages as in Fig. 8a but a different set of equivalent star voltages.

so chosen (Fig. 8b) as to correspond with one of the vertices of the voltage polygon. Under this condition the star voltages are expressed directly in terms of the line-voltages. Thus from the vector diagram Fig. 8b we have:

$$\left. \begin{array}{ll} \mathbf{E}_{01} = 0 & (a) \\ \mathbf{E}_{02} = \mathbf{E}_{12} & (b) \\ \mathbf{E}_{03} = \mathbf{E}_{12} + \mathbf{E}_{23} & (c) \end{array} \right\} \quad \left. \begin{array}{ll} \mathbf{E}_{04} = \mathbf{E}_{12} + \mathbf{E}_{23} + \mathbf{E}_{34} & (d) \\ \mathbf{E}_{05} = -(\mathbf{E}_{56} + \mathbf{E}_{61}) & (e) \\ \mathbf{E}_{06} = -\mathbf{E}_{61} & (f) \end{array} \right\} \quad (21)$$

B. The Equivalent Star Load.—Since we are here dealing with unbalanced polyphase circuits it is necessary to differentiate between the following classes of load equivalence:

- (1) *Unique Equivalence of Polyphase Loads.* Mesh- and star-connected loads are uniquely equivalent if, *only for a given set of impressed line-voltages*, the two loads can be interchanged without affecting either the magnitude or the direction of the vector line-currents.

- (2) *Conditional Equivalence of Polyphase Loads.* Mesh- and star-connected loads are conditionally equivalent if *certain conditions have to be satisfied by the vector line-voltages* before the loads can be interchanged without affecting either the magnitude or the direction of the vector line-currents.
- (3) *Universal Equivalence of Polyphase Loads.* Mesh- and star-connected loads are universally equivalent if, *under any and all conditions of impressed line-voltages*, the two loads can be interchanged without affecting either the magnitude or the direction of the vector line-currents.

Although unique equivalence may be obtained for any polyphase circuit, universal equivalence of loads occurs only in polyphase circuits having not more than three phases.*

We shall therefore simply indicate how a three-phase mesh-connected load may be replaced by a *universally equivalent* three-phase star load and vice versa.

Let the Emfs. across a three-phase mesh load, Fig. 6, be $\mathbf{E}_{ab}$, $\mathbf{E}_{bc}$, $\mathbf{E}_{ca}$. Then the load-currents are:

$$\left. \begin{aligned} \mathbf{I}_{ab} &= \mathbf{E}_{ab}/\underline{Z}_{ab} & (a) \\ \mathbf{I}_{bc} &= \mathbf{E}_{bc}/\underline{Z}_{bc} & (b) \\ \mathbf{I}_{ca} &= \mathbf{E}_{ca}/\underline{Z}_{ca} & (c) \end{aligned} \right\} \quad (22)$$

Applying Kirchhoff's current law to the points a , b , and c , Fig. 6, we have:

$$\left. \begin{aligned} \mathbf{I}_{a1} &= \mathbf{I}_{ca} - \mathbf{I}_{ab} = (\mathbf{E}_{ca}/\underline{Z}_{ca}) - (\mathbf{E}_{ab}/\underline{Z}_{ab}) & (a) \\ \mathbf{I}_{b2} &= \mathbf{I}_{ab} - \mathbf{I}_{bc} = (\mathbf{E}_{ab}/\underline{Z}_{ab}) - (\mathbf{E}_{bc}/\underline{Z}_{bc}) & (b) \\ \mathbf{I}_{c3} &= \mathbf{I}_{bc} - \mathbf{I}_{ca} = (\mathbf{E}_{bc}/\underline{Z}_{bc}) - (\mathbf{E}_{ca}/\underline{Z}_{ca}) & (c) \end{aligned} \right\} \quad (23)$$

Next consider a three-phase star-star circuit (Fig. 4). The line-voltages across the load are $\mathbf{E}_{ab}$, $\mathbf{E}_{bc}$ and $\mathbf{E}_{ca}$. Writing Kirchhoff's Emf. law for the circuits aNb , and bNc we have:

$$\left. \begin{aligned} \mathbf{E}_{ab} &= \underline{Z}_{bN}\mathbf{I}_{N2} - \underline{Z}_{aN}\mathbf{I}_{N1} & (a) \\ \mathbf{E}_{bc} &= \underline{Z}_{cN}\mathbf{I}_{N3} - \underline{Z}_{bN}\mathbf{I}_{N2} & (b) \end{aligned} \right\} \quad (24)$$

Applying Kirchhoff's current law to the point N (assuming no connection between neutrals) we have:

$$\mathbf{I}_{N1} + \mathbf{I}_{N2} + \mathbf{I}_{N3} = 0 \quad (25)$$

Solving Eq. (24a) for $\mathbf{I}_{N1}$ and Eq. (24b) for $\mathbf{I}_{N3}$; substituting these values in Eq. (25) and solving that equation for $\mathbf{I}_{N2}$ we obtain:

$$\mathbf{I}_{N2} = \frac{\mathbf{E}_{ab}\underline{Z}_{cN} - \mathbf{E}_{bc}\underline{Z}_{aN}}{\underline{Z}_{aN}\underline{Z}_{bN} + \underline{Z}_{bN}\underline{Z}_{cN} + \underline{Z}_{cN}\underline{Z}_{aN}} \quad (26)$$

* See author's thesis, "A study of Equivalent Mesh and Star Polyphase Systems," for M.E.E., Cornell University, June 1924.

Now, if the mesh and star three-phase circuits are to be universally equivalent, not only must the line-current I_{b2} (Eq. 23b) be equal to the line-current I_{N2} (Eq. 26) but these currents must be identical for any values of the line-voltages E_{ab} and E_{bc} . Such can be the case if, and only if, Eq. (23b) is *identically* equal to Eq. (26). In other words, the coefficients of the Emfs. E_{ab} and E_{bc} in Eqs. (23b) and (26) must be equal.

Equating these coefficients we have:

$$\left. \begin{aligned} Z_{ab} &= (Z_{aN}Z_{bN} + Z_{bN}Z_{cN} + Z_{cN}Z_{aN})/Z_{cN} & (a) \\ Z_{bc} &= (Z_{aN}Z_{bN} + Z_{bN}Z_{cN} + Z_{cN}Z_{aN})/Z_{aN} & (b) \\ Z_{ca} &= (Z_{aN}Z_{bN} + Z_{bN}Z_{cN} + Z_{cN}Z_{aN})/Z_{bN} & (c) \end{aligned} \right\} \quad (27)^*$$

Eqs. (27) give the impedances of the mesh load in terms of the impedances of the star load. In order to obtain similar expressions for the star-impedances in terms of the mesh-impedances we multiply both sides of Eq. (27a), (27b) and (27c) by Z_{cN} , Z_{aN} , and Z_{bN} respectively. This makes the right-hand sides of the three equations identical. Hence equating the left-hand sides we obtain:

$$Z_{ab}Z_{cN} = Z_{bc}Z_{aN} = Z_{ca}Z_{bN} \quad (28)$$

whence

$$\left. \begin{aligned} Z_{bN} &= Z_{aN}(Z_{bc}/Z_{ca}) & (a) \\ Z_{cN} &= Z_{aN}(Z_{bc}/Z_{ab}) & (b) \end{aligned} \right\} \quad (29)$$

Substituting these values of Z_{bN} and Z_{cN} in Eq. (29a) and solving the resultant expression for Z_{aN} we obtain:

$$\left. \begin{aligned} Z_{aN} &= (Z_{ca}Z_{ab})/(Z_{ab} + Z_{bc} + Z_{ca}) & (a) \\ Z_{bN} &= (Z_{ab}Z_{bc})/(Z_{ab} + Z_{bc} + Z_{ca}) & (b) \\ Z_{cN} &= (Z_{bc}Z_{ca})/(Z_{ab} + Z_{bc} + Z_{ca}) & (c) \end{aligned} \right\} \quad (30)$$

Note that Eqs. (30b) and (30c) were written by cyclic transformation of subscripts. They could, however, be derived as was Eq. (30a).

Sometimes it is more convenient to use admittances instead of impedances. The following expressions are obtained from Eqs. (27) and (30) by simply substituting $Z = 1/Y$:

$$\left. \begin{aligned} Y_{ab} &= (Y_{aN}Y_{bN})/(Y_{aN} + Y_{bN} + Y_{cN}) & (a) \\ Y_{bc} &= (Y_{bN}Y_{cN})/(Y_{aN} + Y_{bN} + Y_{cN}) & (b) \\ Y_{ca} &= (Y_{cN}Y_{aN})/(Y_{aN} + Y_{bN} + Y_{cN}) & (c) \end{aligned} \right\} \quad (27')$$

$$\left. \begin{aligned} Y_{1N} &= (Y_{ab}Y_{ca} + Y_{bc}Y_{ab} + Y_{ca}Y_{bc})/Y_{bc} & (a) \\ Y_{2N} &= (Y_{ab}Y_{ca} + Y_{bc}Y_{ab} + Y_{ca}Y_{bc})/Y_{ca} & (b) \\ Y_{3N} &= (Y_{ab}Y_{ca} + Y_{bc}Y_{ab} + Y_{ca}Y_{bc})/Y_{ab} & (c) \end{aligned} \right\} \quad (30')$$

* Eq. (27c) was written by cycle substitution of subscripts.

It must be here recalled that the equivalent star-star circuit has no neutral connection. In other words, Z_{0N} (Fig. 4) is infinite.

9. Universal Equivalence Is Independent of the Choice of the Source Neutral 0. — It is stated in Art. 8A that any point 0 (Figs. 8) could be taken as the neutral point of the star source which is equivalent to the given mesh source. Evidently then there exists an infinite number of star-connected sources each of which is equivalent to the given mesh-connected source. Now with a given set of line-voltages and load-impedances the line-currents are determinate. Hence it becomes necessary to show that the line-currents of the equivalent star-star circuit are also determinate and that these line-currents are independent of the choice of the source neutral 0.

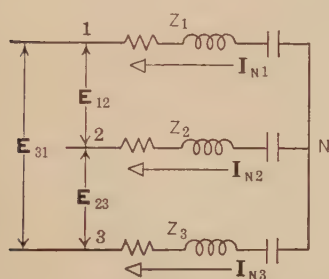


Fig. 9a

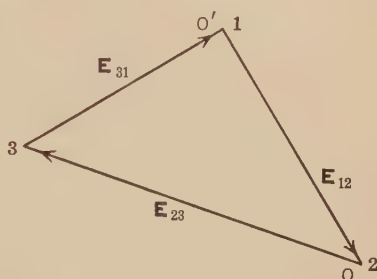


Fig. 9b

FIG. 9a. A three-phase, unbalanced, star load.

FIG. 9b. Vector diagram of the line-voltages impressed on the star load in Fig. 9a.

Proof. — Referring to Figs. 9 let the given line-voltages be E_{12} , E_{23} , and E_{31} , let the load-impedances be Z_1 , Z_2 and Z_3 and the line-currents be I_{N1} , I_{N2} and I_{N3} . Let the points 0 and O' (Fig. 9b) be taken respectively as the neutral points of the equivalent star voltages. This gives:

$$\left. \begin{aligned} E_{01} &= -E_{12} & (a) & & E_{O'1} &= 0 & (a') \\ E_{02} &= 0 & (b) & & E_{O'2} &= E_{12} & (b') \\ E_{03} &= E_{23} & (c) & & E_{O'3} &= -E_{31} & (c') \end{aligned} \right\} \quad (31)$$

The line-currents in each case would be (see Eqs. 12).

$$\left. \begin{aligned} I_{N1} &= (E_{01} - E_{0N}) Y_1 & (a) & & I'_{N1} &= (E_{O'1} - E_{O'N}) Y_1 & (a') \\ I_{N2} &= (E_{02} - E_{0N}) Y_2 & (b) & & I'_{N2} &= (E_{O'2} - E_{O'N}) Y_2 & (b') \\ I_{N3} &= (E_{03} - E_{0N}) Y_3 & (c) & & I'_{N3} &= (E_{O'3} - E_{O'N}) Y_3 & (c') \end{aligned} \right\} \quad (32)$$

We want to show that the line-currents are not affected by the choice of origin — in other words, that $I_{N1} = I'_{N1}$, $I_{N2} = I'_{N2}$ and $I_{N3} = I'_{N3}$ or that (see Eqs. 32):

$$\left. \begin{aligned} E_{01} - E_{0N} &= E_{O'1} - E_{O'N} & (a) \\ E_{02} - E_{0N} &= E_{O'2} - E_{O'N} & (b) \\ E_{03} - E_{0N} &= E_{O'3} - E_{O'N} & (c) \end{aligned} \right\} \quad (33)$$

If the neutrals 0 and N are isolated, then $Y_{0N} = 0$. Hence the values of E_{0N} and $E_{0'N}$ (see Eq. 14) are:

$$\left. \begin{aligned} E_{0N} &= (E_{01}Y_1 + E_{02}Y_2 + E_{03}Y_3)/\sum Y_s \quad (a) \\ E_{0'N} &= (E'_{01}Y_1 + E'_{02}Y_2 + E'_{03}Y_3)/\sum Y_s \quad (b) \end{aligned} \right\} \quad (34)$$

Substituting in Eqs. (33) and (34) the values of the star voltages as given in Eqs. (31) we obtain:

$$\left. \begin{aligned} E_{01} - E_{0N} &= \left[- (E_{12} + E_{23})Y_3 - E_{12}Y_2 \right] / \sum Y_s, \\ &= (E_{31}Y_3 - E_{12}Y_2) / \sum Y_s \quad (a) \\ E_{0'1} - E_{0'N} &= (E_{31}Y_3 - E_{12}Y_2) / \sum Y_s \quad (b) \end{aligned} \right\} \quad (35)$$

But Eqs. (35a) and (35b) are identical. Hence the currents I_{N1} and I'_{N1} are identical. By a similar reasoning it can be shown that I_{N2} and I'_{N3} are identical respectively with I'_{N2} and I'_{N3} , and that if a third neutral 0'' is taken at any point the line-currents I''_{N1} , I''_{N2} and I''_{N3} will also be identical with the currents I_{N1} , I_{N2} and I_{N3} respectively.

10. Current and Voltage Relations in Unbalanced Polyphase Circuits Subjected to Non-Sine Voltages. — We have seen, in the study of non-sine waves, that each harmonic produces its own current as though it were acting alone. Hence the solutions outlined above for sine waves can be used for non-sine waves provided that the equations are applied separately for each harmonic. A numerical problem (10a at the end of this chapter) is solved in order to illustrate the general method of attack.

11. Power Supplied to the Load of an Unbalanced Polyphase Circuit. — (a) *Average Power.* — The average power consumed in each phase of the load in an unbalanced circuit may be computed either by using the expression $P_p = R_p I_p^2$ or by using the relation $P_p = E_p I_p \cos \phi_p = \mathbf{E} \mathbf{I}$ (real). In the first expression R_p is the load resistance per phase and I_p is the effective value of the current per phase flowing through that resistance. In the second expression, E_p is the effective voltage measured across the load through which the effective current I_p flows; and ϕ_p is the phase angle between E_p and I_p . We stress this fact because the student is apt to multiply the line-voltage by the line-current and call this product the power per phase. The power per phase is the Emf. *per phase* multiplied by the current *per phase*.

Let $P_1, P_2, \dots, P_m$ be the average power in phases 1, 2, . . . m of the load. Then:

For star loads (see Figs. 4 and 5):

$$\left. \begin{aligned} P_1 &= R_{aN} I_{N1}^2 = E_{aN} I_{N1} \cos \phi_1 & (a) \\ P_2 &= R_{bN} I_{N2}^2 = E_{bN} I_{N2} \cos \phi_2 & (b) \\ &\vdots & \\ P_m &= R_{m'N} I_{Nm'}^2 = E_{m'N} I_{Nm'} \cos \phi_m & (m) \end{aligned} \right\} \quad (36)$$

For mesh loads (see Figs. 6 and 7):

$$\left. \begin{aligned} P_1 &= R_{ab} I_{ab}^2 = E_{ab} I_{ab} \cos \phi_1 & (a) \\ P_2 &= R_{bc} I_{bc}^2 = E_{bc} I_{bc} \cos \phi_2 & (b) \\ &\vdots & \\ P_m &= R_{L'm'} I_{L'm'}^2 = E_{L'm'} I_{L'm'} \cos \phi_m & (m) \end{aligned} \right\} \quad (37)$$

The total power supplied to the load (mesh or star) is:

$$P = P_1 + P_2 + \dots + P_m \quad (38)$$

(b) *Apparent Power*. — The apparent power per phase of the load may be determined by taking the product of the current flowing through that phase by the voltage across it. Thus if P_{a1} , P_{a2} . . . P_{am} is the apparent power in phases 1, 2 and m of the load, then:

For star loads (see Figs. 4 and 5)

$$\left. \begin{aligned} P_{a1} &= E_{aN} I_{N1} & (a) \\ P_{a2} &= E_{bN} I_{N2} & (b) \\ &\vdots & \\ P_{am} &= E_{m'N} I_{Nm'} & (m) \end{aligned} \right\} \quad (39)$$

For mesh loads (see Figs. 6 and 7)

$$\left. \begin{aligned} P_{a1} &= E_{ab} I_{ab} & (a) \\ P_{a2} &= E_{bc} I_{bc} & (b) \\ &\vdots & \\ P_{am} &= E_{l'm'} I_{l'm'} & (m) \end{aligned} \right\} \quad (40)$$

The total apparent power is:

$$P_a = P_{a1} + P_{a2} + \dots + P_{am} \quad (41)$$

(c) *Power Factor*. — Each phase of the load in an unbalanced m -phase circuit has its own power factor, which may be defined as:

$$\text{P.f.}_n = P_n / P_{an} \quad (42)$$

where $P.f._n$, P_n and P_{an} are respectively the power factor, the average power, and the apparent power in the n th phase of an m -phase unbalanced circuit.

The A.I.E.E. defines the equivalent power factor of an unbalanced polyphase load as follows:

$$P.f. = \sum P_n / \sum P_{an} \quad (43)$$

12. Power Consumed in the Lines of an Unbalanced Polyphase Circuit.—(a) *Average Power.*—Figs. 4 to 7 show that the lines through which power is conveyed to the load possess impedances Z_{a1} , $Z_{b2} \dots Z_{m'm}$. The average power consumed in each of the lines 1, 2 $\dots m$ (Figs. 4 to 7) is:

$$\left. \begin{array}{lll} \text{mesh} & \text{star} & \\ P'_1 = R_{a1} I_{a1}^2 & = R_{a1} I_{N1}^2 & (a) \\ P'_2 = R_{b2} I_{b2}^2 & = R_{b2} I_{N2}^2 & (b) \\ \vdots & \vdots & \\ P'_m = R_{m'm} I_{m'm}^2 & = R_{m'm} I_{Nm}^2 & (m) \end{array} \right\} \quad (44)$$

The total average power consumed by the line is:

$$P' = P_1 + P_2 + \dots + P_m \quad (45)$$

(b) *Apparent Power.*—The apparent power in each line of a polyphase circuit may be computed by multiplying the voltage ($Z_l I_l$) across the line-impedance by the line-current. Thus the apparent power in each of the lines 1, 2 $\dots n$ (Figs. 4 to 7) is:

$$\left. \begin{array}{lll} \text{mesh} & \text{star} & \\ P'_{a1} = Z_{a1} I_{a1}^2 & = Z_{a1} I_{N1}^2 & (a) \\ P'_{a2} = Z_{b2} I_{b2}^2 & = Z_{b2} I_{N2}^2 & (b) \\ \vdots & \vdots & \\ P'_{am} = Z_{m'm} I_{m'm}^2 & = Z_{m'm} I_{Nm}^2 & (m) \end{array} \right\} \quad (46)$$

The total apparent power supplied to the lines is:

$$P'_a = P'_{a1} + P'_{a2} + \dots + P'_{am} \quad (47)$$

13. Power Supplied by an Unbalanced m -Phase Source.—According to the principle of the conservation of energy the total power supplied by the source is equal to the total power consumed by the line and load; or:

$$\left. \begin{array}{l} P_s = P + P' \quad (a) \\ P_{as} = P_a + P'_a \quad (b) \end{array} \right\} \quad (48)$$

where P_s and P_{as} are respectively the average and apparent power supplied by the polyphase source. The *equivalent power factor* of the polyphase source is:

$$\text{P.f.}_s = P_s/P_{as} \quad (49)$$

14. Power Supplied and Consumed in Polyphase Circuits Having Non-Sine Sources. — When the Emf. sources contain harmonics, the procedure of computing the power supplied by each harmonic and consumed by the load and line is as outlined in Arts. 12 to 13. It should be here remembered that each harmonic delivers its own power as though it were acting alone.

PROBLEMS

1. It is required to transform a three-phase balanced source into a six-phase star-connected balanced source.

- Show, by diagrams of connections similar to those given in Figs. 1a and 1c how such a transformation may be effected.
- Give vector diagrams of the phase- and line-voltages of the six-phase star-connected source.
- Express the star and line-voltages in terms of the given phase-voltages of the three-phase balanced source.

2. Draw: (a) A two-phase unbalanced circuit belonging to type *A* (non-inter-connected).

(b) A two-phase unbalanced circuit belonging to type *B* (star-star):

(1) neutrals connected.

(2) neutrals isolated.

(c) A two-phase unbalanced circuit belonging to type *E* (star-mesh).

3. Let the Emfs. in a two-phase circuit be $E_{1/1} = 100 \angle 0^\circ$ and $E_{2/2} = 110 \angle 90^\circ$. Let the impedance per phase be $Z = 5 + j8$. Compute the vector currents supplied by each of the phases when the phases are not inter-connected.

4a. Let the star voltages and star impedances (load + line) of a three-phase unbalanced circuit, with isolated neutrals, be:

$$\begin{array}{ll} \mathbf{E}_{01} = 1330 + j0 & \mathbf{Z}_1 = 5.5 + j4 \\ \mathbf{E}_{02} = -665 - j1150 & \mathbf{Z}_2 = 5.5 + j4 \\ \mathbf{E}_{03} = -665 + j1150 & \mathbf{Z}_3 = 2.5 + j4 \end{array}$$

It is required to determine the vector line-currents.

Solution. —

$$\begin{array}{l} \mathbf{Y}_1 = (1/\mathbf{Z}_1) = 0.119 - j0.0867 \\ \mathbf{Y}_2 = (1/\mathbf{Z}_2) = 0.119 - j0.0867 \\ \mathbf{Y}_3 = (1/\mathbf{Z}_3) = 0.1125 - j0.18 \end{array}$$

whence:

$$\mathbf{E}_s \mathbf{Y}_s = (1330 + j0) (0.119 - j0.0867) + (-665 - j1150) \times (0.119 - j0.0867) + (-665 + j1150) (0.1125 - j0.18) = 112.5 + j55$$

$$\mathbf{Y}_s = (0.119 + 0.119 + 0.1125) - j(0.0867 + 0.0867 + 0.18) = 0.3505 - j0.3534$$

$$\mathbf{E}_{0N} = (\sum \mathbf{E}_s \mathbf{Y}_s / \sum \mathbf{Y}_s) = (112.5 + j55) / (0.3505 - j0.3534) = 82 + j242$$

$$\mathbf{I}_{N1} = (\mathbf{E}_{01} - \mathbf{E}_{0N}) \mathbf{Y}_1 = 128 - j137$$

$$\mathbf{I}_{N2} = (\mathbf{E}_{02} - \mathbf{E}_{0N}) \mathbf{Y}_2 = -208 - j103$$

$$\mathbf{I}_{N3} = (\mathbf{E}_{03} - \mathbf{E}_{0N}) \mathbf{Y}_3 = 80 + j240$$

Then

$$\mathbf{I}_{N1} + \mathbf{I}_{N2} + \mathbf{I}_{N3} = 0 + j0$$

This furnishes a check on the accuracy of the numerical results.

4b. Show that a two-phase star-star circuit with the neutrals 0 and N isolated amounts to a single circuit having an Emf. $\mathbf{E}_{12} = \mathbf{E}_{2/2} - \mathbf{E}_{1/1}$ and an impedance $\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2$.

Give the expression for the current flowing through such a circuit in terms of the phase Emfs. and phase impedances.

4c. Apply Eqs. (11), (12), (13), and (14), to a two-phase star-star circuit in which the neutrals 0 and N are connected through an impedance $\mathbf{Z}_{0N}$.

4d. The phase Emfs. of a two-phase star-star circuit are $\mathbf{E}_{01} = 110 + j0$ and $\mathbf{E}_{02} = 0 + j110$. The load impedances are $\mathbf{Z}_1 = 10 + j7$; $\mathbf{Z}_2 = 5 + j6$. The conductor connecting the neutrals 0 and N has an impedance $\mathbf{Z}_{0N} = 7 + j4$.

(a) Compute the currents $\mathbf{I}_{N1}$, $\mathbf{I}_{N2}$ and $\mathbf{I}_{N0}$.

(b) Draw a vector diagram showing (1) the impressed phase-voltages, (2) the line-currents and (3) the drops across the impedances.

(c) Show by the vector diagram that the drops across the impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are equal respectively to $\mathbf{E}_{01} - \mathbf{E}_{0N}$ and $\mathbf{E}_{02} - \mathbf{E}_{0N}$.

4e. An annealing furnace consists of three resistors, 5 ohms each, connected in Y. The furnace is supplied by a T-circuit having a line-voltage of 110 volts.

(a) Write the complex expression for each of the phase-voltages, assuming the teaser* voltage to be $0 - j95.5$ volts.

(b) Compute the currents flowing in each phase of the load.

(c) Draw vector diagram of the phase-voltages and currents.

4f. A Y-connected load consists of two impedances $\mathbf{Z}_{1N} = \mathbf{Z}_{2N} = \mathbf{Z}$ and a third impedance $\mathbf{Z}_{3N} = \mathbf{Z}/S$ where S is a numeric. This load is subjected to a balanced three-phase star-connected source in which $\mathbf{E}_{01}$ leads $\mathbf{E}_{02}$ and $\mathbf{E}_{03}$ by 120° and 240° , respectively. If the star voltages have effective values E , find the complex expression for $\mathbf{E}_{0N}$ in terms of E , $\mathbf{Z}$ and S .

4g. Given the following star voltages and star admittances (load + line):

$\mathbf{E}_{01} = 100 + j20$	$\mathbf{Y}_1 = 0.3 + j0.4$
$\mathbf{E}_{02} = -50 - j70$	$\mathbf{Y}_2 = 0.5 + j0.8$
$\mathbf{E}_{03} = -70 + j80$	$\mathbf{Y}_3 = 0.2 - j0.6$
$\mathbf{E}_{04} = 50 - j60$	$\mathbf{Y}_4 = 1.3 - j2.5$

Compute the line-currents.

4h. In order to determine the equivalent star impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z}_3$ in a star-star unbalanced circuit, the neutral N of the load is grounded. Oscillograms are then taken of the currents flowing through the line and the voltage between each line and the ground. The vectors representing these oscillograms are:

$\mathbf{E}_{1N} = 8000 + j22,000$	$\mathbf{I}_{N1} = 100 + j200$
$\mathbf{E}_{2N} = 12,000 - j20,000$	$\mathbf{I}_{N2} = 200 - j250$
$\mathbf{E}_{3N} = -25,000 - j5000$	$\mathbf{I}_{N3} = -300 + j50$

Determine the equivalent star impedances of the circuit.

* In a T-connection the term teaser is applied to the transformer which gives an Emf. equal to $(\sqrt{3}/2)E$, and the term "main transformer" is used to refer to that transformer whose total Emf. is E . This transformer is provided with the middle tap 0 shown in Fig. 2a.

5a. The line-voltages and star impedances (line + load) of a three-phase unbalanced mesh-star circuit are:

$$\begin{array}{ll} \mathbf{E}_{12} = 150 + j80 & \mathbf{Z}_1 = 10 + j5 \\ \mathbf{E}_{23} = -90 + j100 & \mathbf{Z}_2 = 6 + j7 \\ \mathbf{E}_{31} = -60 - j180 & \mathbf{Z}_3 = 8 + j9 \end{array}$$

It is required to compute the line (or load) currents.

Solution. — Using Eqs. (15) and (16) we have:

$$\begin{array}{ll} 150 + j80 = \mathbf{I}_{N2}(6 + j7) - \mathbf{I}_{N1}(10 + j5) & (a) \\ -90 + j100 = \mathbf{I}_{N3}(8 + j9) - \mathbf{I}_{N2}(6 + j7) & (b) \\ \mathbf{I}_{N1} + \mathbf{I}_{N2} + \mathbf{I}_{N3} = 0 & (c) \end{array}$$

Solving Eqs. (a) and (b) for $\mathbf{I}_{N1}$ and $\mathbf{I}_{N3}$ we obtain:

$$\begin{array}{ll} \mathbf{I}_{N1} = -15.2 - j0.4 + (0.756 + j0.32) \mathbf{I}_{N2} & (d) \\ \mathbf{I}_{N3} = 1.3 + j11.7 + (0.765 + j0.015) \mathbf{I}_{N2} & (e) \end{array}$$

Substituting Eqs. (d) and (e) in (c) and solving for $\mathbf{I}_{N2}$ we have:

$$\mathbf{I}_{N2} = (13.9 - j11.3)/(2.52 + j0.335) = 4.8 - j5.1$$

Substituting this value of $\mathbf{I}_{N2}$ in Eqs. (d) and (e) we obtain:

$$\begin{array}{l} \mathbf{I}_{N1} = -9.9 - j2.7 \\ \mathbf{I}_{N3} = 5.1 + j7.8 \end{array}$$

Adding $\mathbf{I}_{N1} + \mathbf{I}_{N2} + \mathbf{I}_{N3} = 0$ which furnishes a check on the numerical accuracy of the work.

5b. The line-voltages and star impedances of a quarter phase mesh-star circuit are:

$$\begin{array}{ll} \mathbf{E}_{12} = 220 + j0 & \mathbf{Z}_1 = 3 + j4 \\ \mathbf{E}_{23} = 0 + j220 & \mathbf{Z}_2 = 4 + j5 \\ \mathbf{E}_{34} = -220 + j0 & \mathbf{Z}_3 = 5 + j6 \\ \mathbf{E}_{41} = 0 - j220 & \mathbf{Z}_4 = 6 + j7 \end{array}$$

(a) Compute the line- (or load-) currents. (*Note.* — The sum of these currents must be equal to zero.)

(b) Draw a vector diagram showing: (1) line-voltages, (2) Emf. drops across the impedances, (3) line-currents.

6a. A delta-connected load consists of three resistances $R_{ab} = 6$, $R_{bc} = 8$ and $R_{ca} = 10$ ohms. The line-impedances are negligible. The line-voltages are 110 volts each. Moreover, $\mathbf{E}_{12}$ leads $\mathbf{E}_{23}$ and $\mathbf{E}_{31}$ by 120° and 240° , respectively. Taking the vector voltage $\mathbf{E}_{12} = 0 + j110$ volts,

(a) Write the complex expressions for $\mathbf{E}_{23}$ and $\mathbf{E}_{31}$.

(b) Compute the load-currents.

(c) Determine the line-currents.

6b. A 220-volt three-phase balanced source supplies an unbalanced mesh load having impedances $\mathbf{Z}_{ab} = 4 - j5$, $\mathbf{Z}_{bc} = 5 + j6$ and $\mathbf{Z}_{ca} = 6 + j7$. Assuming that the line-impedances are $\mathbf{Z}_{a1} = \mathbf{Z}_{b2} = \mathbf{Z}_{c3} = 0.1 + j0.2$ and that the line-voltages are:

$$\mathbf{E}_{12} = 220 + j0; \quad \mathbf{E}_{23} = -110 + j191; \quad \mathbf{E}_{31} = -110 - j191$$

Compute the load- and line-currents.

7a. A star-mesh circuit has the load and line characteristics specified in problem 6b. The star voltages are:

$$\mathbf{E}_{01} = 100 + j50; \quad \mathbf{E}_{02} = 60 - j90; \quad \mathbf{E}_{03} = -70 + j80$$

Compute the load- and line-currents.

7b. A quarter-phase source supplies a balanced mesh load such that the line-currents are I amperes each. The branch 2-3 of the load is suddenly disconnected. What will be the effective values of the steady line-currents

$$I_{a1}, I_{a2} \text{ and } I_{a4}$$

in terms of the former line-currents I ?

8a. A six-phase rotary converter is symmetrically built so that the vector diagram for the line-voltages forms a hexagon. Assuming the sequence of the vectors to be $\mathbf{E}_{12}, \mathbf{E}_{23} \dots \mathbf{E}_{61}$,

- (a) Replace these mesh voltages by an equivalent star taking the point 2 as the common point 0 of the star voltages.
- (b) Express the star voltages in terms of the given mesh voltages.

8b. A delta-connected load consists of three impedances $Z_{12} = 2 + j4$, $Z_{23} = 3 - j2$, and $Z_{31} = 2 + j$ ohms. What is the equivalent Y-connected load?

8c. Three admittances $Y_{1N} = 0.5 - j0.7$, $Y_{2N} = 0.8 + j0.5$ and $Y_{3N} = 0.4 + j0.3$ mhos are connected in star. What is the equivalent delta-connected load?

8d. A Y-connected load has (line + load) impedances $Z_1 = 5 + j6$, $Z_2 = 3 + j2$, and $Z_3 = 2 + j1$. The line-voltages are $\mathbf{E}_{12} = -60 + j180$, $\mathbf{E}_{23} = 200 - j75$, and $\mathbf{E}_{31} = -140 - j105$.

- (a) Determine a set of star voltages that are equal to the given mesh voltages, taking point 1 as the zero point.
- (b) Compute the line-currents in complex form.
- (c) Draw vector diagram showing:
 - (1) Line-voltages.
 - (2) Line-currents.
 - (3) Voltages across the load.
 - (4) Voltage $\mathbf{E}_{0N}$ between neutrals.

8e. A 220-volt Y-source is connected to a star load having admittances $Y_1 = 0.3 + j0.4$, $Y_2 = 0.4 - j0.5$, $Y_3 = 0.5 + j0.6$. The neutral points of the load and source are connected to each other by an admittance $Y_{0N} = 0.1 + j0.2$. Taking the line-voltage $\mathbf{E}_{12} = 220 + j0$,

- (a) Give the complex expressions for $\mathbf{E}_{23}$ and $\mathbf{E}_{31}$.
- (b) Let the point 3 be the neutral point of the equivalent star source. Give the values of $\mathbf{E}_{01}$, $\mathbf{E}_{02}$ and $\mathbf{E}_{03}$ in terms of the given line-voltages.
- (c) Compute the line-currents of the equivalent star-star circuit.

8f. Solve problems 5a, 6a, and 7a by converting the circuits to equivalent star-star circuits and check the results so obtained with your previous results.

9. Convert the line-voltages in problem 5a to equivalent star voltages:

- (a) By taking the point 1 as the neutral 0 of the source.
- (b) By taking the point 2 as the neutral 0 of the source.
- (c) By taking a point 3 as the neutral 0 of the source.
- (d) By taking a point midway between the points 2 and 3 as the neutral 0 of the source.

Now compute the line-currents I_{N1} , I_{N2} and I_{N3} for each of the four equivalent star-star circuits and convince yourself that these currents are not altered by the choice of the source neutral.

10a. A two-phase source is transformed to a quarter-phase source as shown in Fig. 1c by connecting the mid-points 0 of the two phases. The instantaneous values of the Emfs. of the two phases are respectively:

$$\begin{aligned} e_{1/1} &= 220 \sin \omega_1 t + 55 \sin 3 \omega_1 t \\ e_{2/2} &= 220 \sin (\omega_1 t + 90^\circ) + 55 \sin 3 (\omega_1 t + 90^\circ) \end{aligned}$$

The resulting star source is impressed on a four-phase star load such that the combined line and load admittances, at fundamental frequency, are:

$$\begin{aligned} Y_1 &= 0.5 + j0.1 & Y_3 &= 0.6 - j0.6 \\ Y_2 &= 0.4 + j0.1 & Y_4 &= 0.7 - j1.2 \end{aligned}$$

- Give the admittances for the third harmonic frequency.
- Express the star voltages due to the fundamental and the third harmonic in the complex form.
- Determine the complex values of the line-currents due to the fundamental and to the third harmonic.
- Give the instantaneous values of the non-sine line-currents.
- Give the instantaneous values of the non-sine star voltages.
- Compute the effective values of the line-currents.

Solution. —

- (a) The values of the admittances for the third harmonic are:

$$\begin{aligned} Y'_{11} &= 0.5 + j0.3 & Y'_{33} &= 0.6 - j0.2 \\ Y'_{22} &= 0.4 + j0.3 & Y'_{44} &= 0.7 - j0.4 \end{aligned}$$

- (b) For the fundamental

$$\begin{aligned} E_{1/1} &= 155.6 \angle 0 \\ E_{2/2} &= 155.6 \angle 90^\circ \end{aligned}$$

Hence (see Fig. 1d):

$$\begin{aligned} E_{01} &= 77.8; E_{01'} = -77.8 \\ E_{02} &= j77.8; E_{02'} = -j77.8 \end{aligned}$$

- (c) For the fundamental

$$\begin{aligned} E_{ON} &= \left(\sum E_s Y_s / \sum Y_s \right) \\ &= -39 - j14.3 \end{aligned}$$

- (b) For the third harmonic

$$\begin{aligned} E'_{1/1} &\times 38.9 \angle 0 \\ E'_{2/2} &= 38.9 \angle 270^\circ \end{aligned}$$

Hence (see Fig. 1d):

$$\begin{aligned} E'_{01} &= 19.45; E'_{01'} = -19.45 \\ E'_{02} &= -j19.45; E'_{02'} = j19.45 \end{aligned}$$

- (c) For the third harmonic

$$\begin{aligned} E'_{ON} &= \left(\sum E'_s Y'_s / \sum Y'_s \right) \\ &= 5.3 + j7.7 \pi 6 \end{aligned}$$

Applying Eqs. (12) we have:

$$\begin{aligned} I_{N1} &= 57 + j18.8 = 60 \angle 18^\circ 10' \\ I_{N2} &= 6.5 + j40.7 = 41 \angle 81^\circ \\ I_{N3} &= -14.7 + j31.8 = 34.6 \angle 115^\circ \\ I_{N4} &= -48.8 - j91.3 = 103.5 \angle -118^\circ \end{aligned}$$

Applying Eqs. (12) we have:

$$\begin{aligned} I'_{N1} &= 9.19 + j0.72 = 9.22 \angle 4^\circ 20' \\ I'_{N2} &= 5.83 - j12.21 = 13.5 \angle -64^\circ 20' \\ I'_{N3} &= -16.26 + j0.71 = 16.3 \angle 177^\circ 20' \\ I'_{N4} &= 1.24 + j10.78 = 10.8 \angle 83^\circ 20' \end{aligned}$$

It will be noted that the sum of the line-currents for both the fundamental and the third harmonic are, respectively, equal to zero. It will be wrong, however, to add the corresponding vector line-currents of the fundamental and third harmonic because the two waves do not have the same frequency.

$$\begin{aligned} (d) \quad i_{N1} &= 85 \sin (\omega_1 t + 18^\circ 10') + 13.02 \sin 3 (\omega_1 t + 1^\circ 27') \\ i_{N2} &= 58 \sin (\omega_1 t + 81^\circ) + 19.1 \sin 3 (\omega_1 t - 21^\circ 27') \\ i_{N3} &= 48.9 \sin (\omega_1 t + 115^\circ) + 23.1 \sin 3 (\omega_1 t + 59^\circ 6') \\ i_{N4} &= -146.3 \sin (\omega_1 t - 118^\circ) + 15.25 \sin 3 (\omega_1 t + 27^\circ 47') \end{aligned}$$

$$\begin{aligned}
 (e) \quad e_{01} &= 110 \sin \omega_1 t + 27.5 \sin 3 \omega_1 t \\
 e_{02} &= 110 \sin (\omega_1 t + 90^\circ) + 27.5 \sin 3 (\omega_1 t + 90^\circ) \\
 e_{03} &= 110 \sin (\omega_1 t + 180^\circ) + 27.5 \sin 3 (\omega_1 t + 180^\circ) \\
 e_{04} &= 110 \sin (\omega_1 t - 90^\circ) + 27.5 \sin 3 (\omega_1 t - 90^\circ)
 \end{aligned}$$

$$\begin{aligned}
 (f) \quad I_{N1} &= \sqrt{60^2 + 9.22^2} = 60.8 \\
 I_{N2} &= \sqrt{41^2 + 13.5^2} = 43.2 \\
 I_{N3} &= \sqrt{34.6^2 + 16.3^2} = 38.3 \\
 I_{N4} &= \sqrt{103.5^2 + 10.8^2} = 104
 \end{aligned}$$

10b. The phase-voltages of a Y-connected source are:

$$\begin{aligned}
 e_{01} &= 90 \sin \omega_1 t + 30 \sin 5 \omega_1 t \\
 e_{02} &= 90 \sin (\omega_1 t - 120^\circ) + 30 \sin 5 (\omega_1 t - 120^\circ) \\
 e_{03} &= 90 \sin (\omega_1 t + 120^\circ) + 30 \sin 5 (\omega_1 t + 120^\circ)
 \end{aligned}$$

This source is impressed on a three-phase star load such that the combined line and load admittances at fundamental frequency are:

$$Y_1 = 0.2 + j0.3; \quad Y_2 = 0.3 - j0.4; \quad Y_3 = 0.4 - j0.5$$

- Give the admittances for the fifth harmonic frequency.
- Express the star voltages due to the fundamental and the fifth harmonic in the complex form.
- Determine the complex values of the line-currents due to the fundamental and to the fifth harmonic.
- Give the instantaneous values of the non-sine line-currents.
- Give the instantaneous values of the non-sine star-voltages.
- Compute the effective values of the line-currents.

11a. Determine the average power supplied to the load of problems 6a, 6b, and 7a.

11b. Compute the apparent power and power factor of the load in problems 6a, 6b, and 7a.

12. Compute the average and apparent power consumed in the line of problem 6b.

13. Determine the average and apparent power supplied by the polyphase sources in problems 4 to 7 inclusive. Also evaluate the P.f. in each case.

14. Determine the average and apparent power supplied by the polyphase sources in problems 10a and 10b. Give the equivalent power factor in each case.

CHAPTER XVII

ELECTRIC TRANSIENTS

PART A — SINGLE-ENERGY TRANSIENTS

1. Introduction. — The term electric transient is applied to such a phenomenon as occurs in an electric circuit only for a certain interval of time after which it either disappears altogether or assumes so small a magnitude that it is hardly measurable. A transient current occurs, for instance, immediately after a voltage is impressed on an inductive or capacitive circuit, or when a circuit is shorted. In general, transients accompany any change either in the Emf. source or in the circuit characteristics or in both.

Transients occur only in circuits where electric energy is or may be stored. Hence no transients occur in a pure resistance circuit. But they do occur in circuits having resistance and inductance or resistance and capacitance because the former have the potentiality of storing energy in an electromagnetic form and the latter in the electrostatic form. Again transient currents occur in circuits consisting of resistance, inductance, and capacitance because these circuits may have energy stored either electrostatically or electromagnetically.

Now in circuits with R and $\mathcal{L}$ or with R and C the *amount* (not the form) of the stored energy may change. Thus, in a circuit with R and $\mathcal{L}$, energy can be stored only in the electromagnetic form and only in the inductance $\mathcal{L}$. Hence a change either in the circuit characteristics or in the impressed Emf. results in a change of the *magnitude* of the stored energy. But, in circuits consisting of R , $\mathcal{L}$, and C , not only the *amount* of the stored energy may be changed but also its *form* may be altered. Thus energy stored in the electromagnetic form may be changed to energy stored electrostatically, or vice-versa. Hence, from an energy viewpoint, transients accompany any change in either the *magnitude or the form* of the energy stored in an electric circuit.

Throughout the foregoing treatment we have dealt with circuits in the steady state, that is, with circuits in which the transient current is either negligible or entirely absent. Our attention will be centered, in this and the succeeding chapter, on circuits in the transient state.

The subject will be divided into two topics:

A. Single-Energy Transients. — These occur in circuits consisting of resistance and inductance or in circuits consisting of resistance and capacitance.

The term "single energy transients" is used because the energy stored in such circuits is of only one form (either electromagnetic or electrostatic).

B. Double-Energy Transients. — These occur in circuits consisting of resistance, inductance, and capacitance, and will be treated in the succeeding chapter. The term "double energy transient" is used because such circuits may have energy stored in two forms (electromagnetically and electrostatically).

2. The Transient State in a Series Circuit Consisting of Resistance R and Inductance $\mathcal{L}$.

A. Physical Considerations.

(1) *Continuous Emf. (Establishing the Circuit).* — Consider the circuit, Fig. 1a. Let the switch S_2 be open and the switch S_1 be suddenly closed. The current i_1 , Fig. 1b, does not reach its final steady value $I = E/R$ at once but increases gradually, following the curve i_1 , until after a time t_1 it assumes a fairly steady value $I = E/R$. Indeed if the current element of an oscillograph is connected, in Fig. 1a, it will register a curve identical with i_1 .

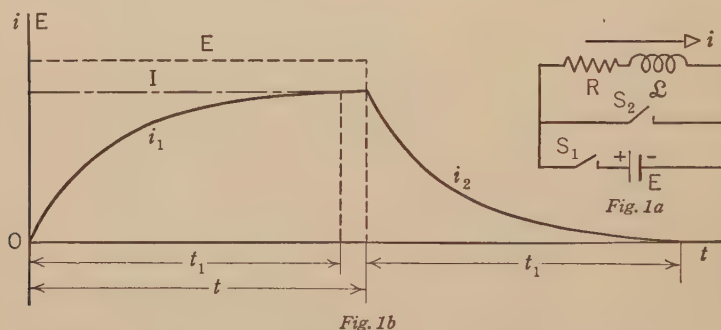


FIG. 1a. A series circuit with R , $\mathcal{L}$, and a continuous Emf. E .

FIG. 1b. Transient currents in the circuit Fig. 1a.

(2) *Continuous Emf. (Short-Circuit).* — With S_1 , Fig. 1a, closed and the current at its steady value I , let the switch S_2 be suddenly closed at an instant $t = t_1$ so as to short-circuit R and $\mathcal{L}$; and let the switch S_1 be simultaneously opened. The current in the circuit does not fall to zero instantly but decreases gradually, following the curve i_2 . After a time t_1 (equal to the time it took the current i_1 to attain its steady value) the current i_2 virtually vanishes, as is shown in Fig. 1b.

The remarkable fact is that although the impressed voltage E across the circuit might be changed instantaneously from zero to E or vice versa (see Fig. 1b) such is not the case with the current. The physical

reason is that the electrical inertia of the circuit (its inductance) prevents the current from changing its value instantly (see p. 52). Hence at the instant of establishing the circuit the current can have no other value except zero. Conversely, at the instant of short-circuit the current can assume no other value except the value it had at that instant. As long as the circuit possesses inductance any change in the magnitude of the current must be gradual, not instantaneous.

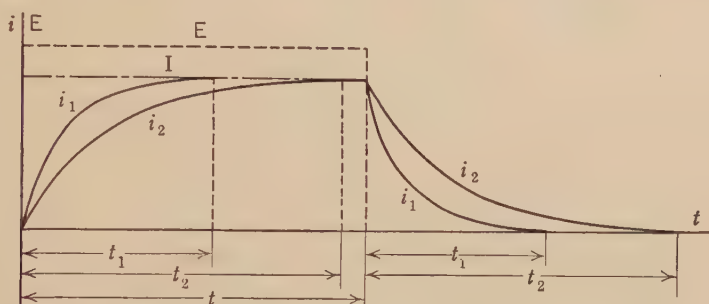


FIG. 2. Effect of the time constant on the transient current in the circuit, FIG. 1a.

(3) *The Time Constant of a Circuit.* — Let two series circuits, with inductances $\mathcal{L}_1$ and $\mathcal{L}_2 > \mathcal{L}_1$, have the same resistance R and let the experiments described in paragraphs (1) and (2) be performed on each of them. The current curves will have the shapes shown in Fig. 2. The current i_2 takes a greater time than i_1 to reach its steady value I after the circuit is established. Moreover i_2 takes a longer time than i_1 to become zero after the circuit is shorted. We thus conclude that with a given resistance, the greater the inductance $\mathcal{L}$ of the circuit the longer the time required to attain the steady state. We shall show later that the rate at which the current increases or decreases in an inductive circuit is not a function of the absolute values of R and $\mathcal{L}$ but rather is a function of the ratio $\mathcal{L}/R$. This ratio is so important in transient phenomena that a name has been given to it. It is known as the *time constant* of the circuit. The reason for calling it the time constant is because $\mathcal{L}/R$ (see Chapter IV) has the physical dimension of time.

(4) *Alternating Sine Emf. (Establishing the Circuit).* — Let a series circuit, consisting of R and $\mathcal{L}$, be subjected to a sine voltage. The shape of the current obtained immediately upon closing the switch S_1 (Fig. 3a) depends altogether upon the instantaneous value e_0 of voltage at the time of closing the switch. If this value of the voltage is such that at the instant $t = 0$, that is, at the instant of closing the switch:

$$e_0 = E_m \sin \alpha = E_m \sin \phi \quad (1)$$

then the transient current would be zero and we have the steady state conditions shown in Fig. 3b. Moreover, the voltage and current waves are so related that:

$$\left. \begin{aligned} e &= E_m \sin (\omega t + \phi) \quad (a) \\ i &= (E_m/Z) \sin \omega t \quad (b) \end{aligned} \right\} \quad (2)$$

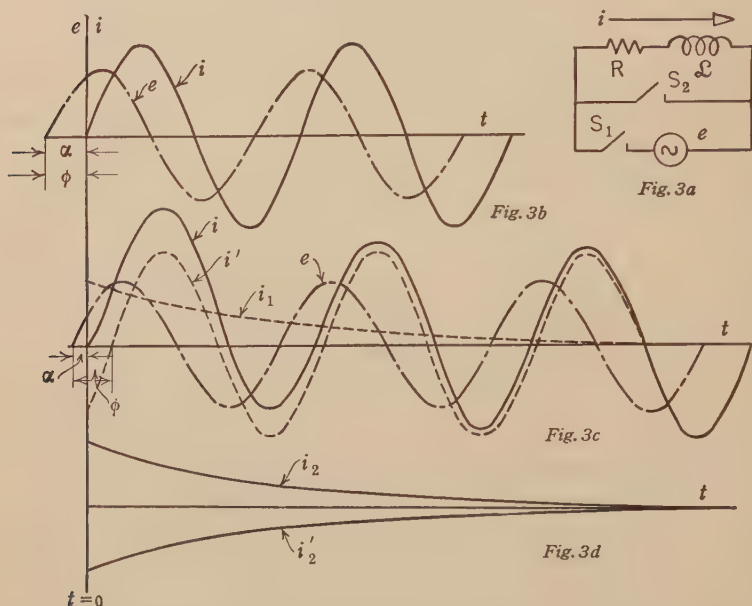


FIG. 3a. A series circuit with R , L , and a sine Emf. e .

FIG. 3b. Emf. and current waves for the steady state in the circuit, Fig. 3a.

FIG. 3c. Emf. and current waves for the transient state when the Emf. is suddenly impressed on the circuit, Fig. 3a.

FIG. 3d. Current curves when the circuit, Fig. 3a, is suddenly short-circuited.

The physical reason for this is that under the conditions specified in Eq. (1) the current would in the steady state be zero at the instant $t = 0$ (see Eq. 2b). Therefore, if the switch S_1 is closed at that instant, there exists no tendency on the part of the current to assume a value other than zero and hence no transient current occurs.

On the other hand, should the switch S_1 be closed at any other instant $t = 0$ such that the instantaneous value e_0 of the voltage wave does not satisfy Eq. (1) then, since under steady conditions the current would assume a value $i_0 \neq 0$ (see curve i' , Fig. 3c), there would appear a transient current i_1 of a magnitude equal to $-i_0$ so that at the instant $t = 0$ the sum of these currents is zero. This transient current persists for a time. An oscillograph connected in the circuit (Fig. 3a) does not

register separately the steady current i' and the transient current i_1 (Fig. 3c) but gives the actual current i such that:

$$i = i' + i_1 \quad (3)$$

Note that i does not satisfy Eqs. (2) until after the transient current i_1 has become negligible. Therefore, the theory outlined in the preceding chapters holds true only for the steady state.

(5) *Alternating Sine Emfs. (Short-Circuit)*. — Let steady conditions in the circuit, Fig. 3a, be established. Then let the switch S_2 be closed and S_1 be simultaneously opened. The current oscillogram which would be obtained under these conditions depends entirely upon the instantaneous value of the current at the time the switch is closed. Three cases will be considered:

- (1) The instantaneous value $i = 0$ when S_2 is closed: Here no short circuit current occurs at all.
- (2) The instantaneous value $i > 0$ when S_2 is closed: The short circuit current under this condition is represented by the curve i_2 , Fig. 3d.
- (3) The instantaneous value $i < 0$ when S_2 is closed. Curve i'_2 (Fig. 3d) represents the short-circuit current which is obtained under this condition

Comparing the curves i_2 , Fig. 1b and Fig. 3d, we conclude that there exists a difference only in the magnitude or direction of the transient currents but not in the general shape of the current curves obtained when continuous or alternating circuits are short-circuited. In the former case a transient always exists but in the latter case the transient may or may not occur. Moreover, the short-circuit transient that does appear with alternating currents has a magnitude which depends upon the instant of short-circuit.

The rate at which the currents i_1 and i_2 , Figs. 3c and 3d, vanish depends altogether upon the time constant of the circuit. If the ratio $\mathcal{L}/R$ is large, these transients may persist for a long time; otherwise they disappear within a small fraction of a second.

(6) *Non-Sine Emf. (Establishing the Circuit and Short-Circuit)*. — It is shown in Chapter XIV that each harmonic behaves as though it were acting alone. No detailed discussion of the transient state with non-sine Emfs. is, therefore, necessary. Suffice it to say that the physical considerations presented in parts (4) and (5) of this article apply to each harmonic considered separately.* Moreover, the resultant cur-

* We assume throughout this treatment that R and $\mathcal{L}$ are constant for all harmonics.

rent is simply the sum of all the steady and all the transient currents due to each harmonic occurring in the non-sine wave.

B. Mathematical Theory.

The voltage e , impressed on a series circuit consisting of R and $\mathcal{L}$, is equal to the sum of the voltage drops across the resistance and inductance. Thus:

$$\begin{aligned} e &= Ri + \mathcal{L} di/dt & (a) \\ \text{or} \quad (di/dt) + (R/\mathcal{L})i &= (1/\mathcal{L})e & (b) \end{aligned} \quad (4)$$

Eq. (4b) is a linear differential equation of the first degree and first order and has the form of Eq. (5.1), Appendix I. Its solution is given by Eq. (5.2), Appendix I. The notations used in Eq. (5.1), Appendix I, bear the following correspondence to those in Eq. (4b):

$$x \sim i; \quad a \sim R/\mathcal{L}; \quad b \sim 1/\mathcal{L}; \quad y \sim e^*$$

The solution of Eq. (4b) is, therefore:

$$i = e^{-Rt/\mathcal{L}} \int (e/\mathcal{L}) e^{Rt/\mathcal{L}} dt + K e^{-Rt/\mathcal{L}} \quad (5)$$

Eq. (5) is the general expression for the current in a circuit consisting of R and $\mathcal{L}$. We shall apply this equation to the interpretation of the current curves shown in Figs. 1 to 3.

(1) *Continuous Emf. (Establishing the Circuit)*. — Let the continuous Emf. applied to the circuit, Fig. 1a, be E . Substituting this value in Eq. (5) and integrating we have:

$$i = (E/R) + K e^{-Rt/\mathcal{L}} \quad (6)$$

The value of the constant K in Eq. (6) may be determined from the physical conditions prevailing in the circuit at the instant $t = 0$. Thus at the instant of closing the switch, that is, at the time† $t = 0$ the current in the circuit is zero. Hence substituting $i = 0$ and $t = 0$ in Eq. (6) and solving for K we have:

$$K = -E/R \quad (7)$$

Substituting this value of K in Eq. (6) we have:

$$i = (E/R) (1 - e^{-Rt/\mathcal{L}}) \quad (8)$$

Eq. (8) gives the value of i , Fig. 1b. It shows that the current is an exponential function of time and that its value increases as time increases until, when $t = \infty$, $I = E/R$, which is Ohm's law.

* The symbol $\sim$ means "corresponds to."

† Throughout this treatment time will be measured from the instant when switching or any change in the circuit conditions occurs; for example, $t = 0$ when S_1 (Fig. 1a) is closed; also $t = 0$ when S_2 is closed, etc.

Although theoretically the steady current $I = E/R$ does not attain its steady value until $t = \infty$, yet, in practice, i_1 reaches this value within a comparatively short time. In some instances the steady value is attained within a fraction of a second after establishing the circuit.

Now in paragraph 3a we introduced the time constant, $\mathcal{L}/R$, of the circuit. We also stated that the rate at which the current changes is a function of the ratio $\mathcal{L}/R$.

We are now in position to give a numerical definition of the time constant. Thus in Eq. (8) let $t = \mathcal{L}/R$. This equation then reduces to:

$$\begin{aligned} i &= (E/R) (1 - e^{-1}) && (a) \\ &= (E/R) (e - 1)/e = I(1.718/2.718) = 0.632 I && (b) \end{aligned} \quad (9)$$

We may then define the time constant as such an interval of time, $t = \mathcal{L}/R$, measured from the instant $t = 0$, that at the end of it the current shall have reached $(e - 1)/e$ or about two thirds of its final steady value. Evidently the greater $\mathcal{L}/R$ is the greater will be the interval of time t required for the transient current to reach two thirds of its steady value and the longer the duration of the transient current (see Fig. 2).

(2) *Continuous Emf. (Short-Circuit)*.—Consider the circuit, Fig. 1a, and let a current $I = E/R$ flow through it. Now let S_2 be closed and S_1 be simultaneously opened. If we consider $t = 0$ when the switching operations take place, then at the time $t = 0$, $E = 0$ and $i = I = E/R$. Substituting these values in Eq. (6) we have:

$$I = 0 + K \quad (10)$$

Substituting this value of K in Eq. (6) and remembering that $E = 0$ at all times after short-circuit we have:

$$i = Ie^{-Rt/\mathcal{L}} \quad (11)$$

Eq. (11) gives the expression for the curve i_2 shown in Fig. 1b. It shows that the short-circuit current is an exponential function of time.

According to Eq. (11), the short-circuit current never dies out because the term on the right-hand side becomes zero only when $t = \infty$. Such, however, is not the case in practice because in some instances the current i does, in the course of a few seconds, assume such small values that it is hardly measurable.

(3) *Alternating Sine Emf. (Establishing the Circuit)*.—Let the voltage e , in Eq. (5), be a sine function of time, or:

$$e = E_m \sin (\omega t + \alpha) \quad (12)$$

where α is such an angle that at the time of switching or of any other change in the circuit conditions (at the time $t = 0$):

$$e_0 = E_m \sin \alpha \quad (13)$$

Substituting in Eq. (5) the value of e given by Eq. (12) we have:

$$i = \epsilon^{-Rt/L} \int (E_m/L) \sin(\omega t + \alpha) \epsilon^{Rt/L} dt + K \epsilon^{-Rt/L} \quad (14)$$

It can be easily shown (see problem 5d, Chapter I) that:

$$\sin(\omega t + \alpha) = \left[\epsilon^{j(\omega t + \alpha)} - \epsilon^{-j(\omega t + \alpha)} \right] / 2j \quad (15)$$

Substituting Eq. (15) in (14) and simplifying we have:

$$i = \frac{(\epsilon^{-Rt/L}) E_m}{2jL} \int \left[\epsilon^{(j\omega + R/L)t} \epsilon^{j\alpha} - \epsilon^{(-j\omega + R/L)t} \epsilon^{-j\alpha} \right] dt + K \epsilon^{-Rt/L} \quad (16)$$

Integrating Eq. (16) we obtain:

$$\left. \begin{aligned} i &= \frac{\epsilon^{-Rt/L} E_m}{2jL} \left[\frac{\epsilon^{(j\omega + R/L)t} \epsilon^{j\alpha}}{(R/L) + j\omega} - \frac{\epsilon^{(-j\omega + R/L)t} \epsilon^{-j\alpha}}{(R/L) - j\omega} \right] + K \epsilon^{-Rt/L} \quad (a) \\ &= \frac{E_m}{2j} \left[\frac{\epsilon^{j(\omega t + \alpha)}}{R + j\omega L} - \frac{\epsilon^{-j(\omega t + \alpha)}}{R - j\omega L} \right] + K \epsilon^{-Rt/L} \quad (b) \end{aligned} \right\} \quad (17)$$

$$\left. \begin{aligned} \text{But} \quad R + j\omega L &= Z \epsilon^{j\phi} \quad (a) \\ \text{and} \quad R - j\omega L &= Z \epsilon^{-j\phi} \quad (b) \end{aligned} \right\} \quad (18)$$

Substituting these values in Eq. (17b) we have:

$$\left. \begin{aligned} i &= \frac{E_m}{2j} \left[\frac{\epsilon^{j(\omega t + \alpha)}}{Z \epsilon^{j\phi}} - \frac{\epsilon^{-j(\omega t + \alpha)}}{Z \epsilon^{-j\phi}} \right] + K \epsilon^{-Rt/L} \quad (a) \\ &= \frac{E_m}{Z} \left[\frac{\epsilon^{j(\omega t + \alpha - \phi)}}{2j} - \frac{\epsilon^{-j(\omega t + \alpha - \phi)}}{2j} \right] + K \epsilon^{-Rt/L} \quad (b) \\ &= (E_m/Z) \sin(\omega t + \alpha - \phi) + K \epsilon^{-Rt/L} \quad (c) \end{aligned} \right\} \quad (19)$$

We shall next determine the value of K for the conditions of establishing the circuit.

At the time the switch S_1 , Fig. 3a, is closed, that is, at the time $t = 0$, the current i is 0. Substituting these values in Eq. (19c) and solving for K we obtain:

$$K = -(E_m/Z) \sin(\alpha - \phi) \quad (20)$$

Substituting Eq. (20) in (19c) we obtain:

$$i = (E_m/Z) [\sin(\omega t + \alpha - \phi) - \sin(\alpha - \phi) \epsilon^{-Rt/L}] \quad (21)$$

Eq. (21) gives the value of the sine current in a circuit consisting of R and $\mathcal{L}$ at any instant after the circuit is established. This current consists of two parts:

- (a) The steady current, namely, $(E_m/Z) \sin(\omega t + \alpha - \phi)$. This term is represented by the curves i' , Figs. 3b and 3c.
- (b) The transient current, namely, $-(E_m/Z) \sin(\alpha - \phi) e^{-Rt/\mathcal{L}}$. This term is represented by the curve i_1 , Fig. 3c, and is absent in Fig. 3b. The reason for the absence of the transient term in Fig. 3b is that $\alpha = \phi$ so that K in Eq. (20) is zero.

(4) *Alternating Sine Emf. (Short-Circuit)*. — Under short-circuit conditions $E_m = 0$ and $i = I_m \sin(\alpha - \phi)$ when $t = 0$. Substituting these values in Eq. (19c) we have:

$$I_m \sin(\alpha - \phi) = 0 + K \quad (22)$$

Substituting this value of K in Eq. (19c) and remembering that for all instants after short-circuit the steady current is zero because $E_m = 0$ we have:

$$i = I_m \sin(\alpha - \phi) e^{-Rt/\mathcal{L}} \quad (23)$$

Eq. (23) shows that the short-circuit current (i_2 , Fig. 3d) is an exponential (not a sine) function of time. Moreover, the value of this current depends altogether upon the instant when short-circuit occurs. If short-circuit occurs at such a value e_0 of the impressed voltage that $\alpha = \phi$, then the transient current i in Eq. (23) reduces to zero. If, however, the short-circuit occurs at any instant such that $\alpha \neq \phi$, then there would be a transient current the initial value of which is dependent upon the difference between α and ϕ . Evidently the maximum transient that could be obtained on short-circuit occurs when $\alpha - \phi = n 90^\circ$, where n is an odd integer.

The transient terms $K e^{-Rt/\mathcal{L}}$ in Eqs. (6) and (19) are identical. Hence what is stated regarding the time constant for continuous Emf. circuits applies equally well to alternating Emf. circuits.

(5) *Non-Sine Emf. (Establishing the Circuit and Short-Circuit)*. — We shall extend the results arrived at in Eqs. (19c), (20), and (23) to cover all harmonics. These equations, for non-sine waves, become, respectively:

General Solution:

$$i = \sum (E_{mn}/Z_n) \sin n(\omega_1 t + \alpha - \phi_n/n) + \sum K_n e^{-Rt/\mathcal{L}} \quad (19'c)$$

Case of Establishing the Circuit:

$$i = \sum (E_{mn}/Z_n) \sin n(\omega_1 t + \alpha_n - \phi_n/n) - \sum \sin(n\alpha_n - \phi_n) e^{-Rt/\mathcal{L}} \quad (20')$$

Case of Short-Circuit:

$$i = \sum I_{mn} \sin (n\alpha_n - \phi_n) e^{-Rt/L} \quad (23')$$

3. The Transient State in a Series Circuit Consisting of Resistance R and Capacitance C .

A. Physical Considerations.

(1) *Continuous Emf. (Establishing the Circuit).* — Fig. 4a represents a series circuit consisting of a resistance R , a capacitance C and a source of continuous Emf. E . Let the switch S_1 be closed while S_2 is open. The current registered by an oscillograph assumes the shape of i_1 , Fig. 4b. The value of i_1 is equal to E/R at the time $t = 0$. This cur-

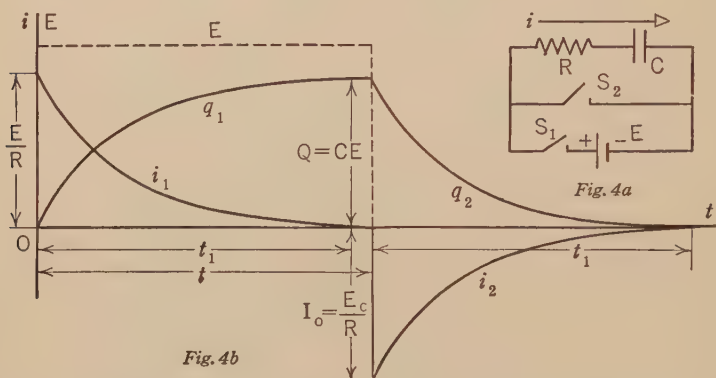


FIG. 4a. A series circuit with R and C and a continuous Emf. E .

FIG. 4b. Transient currents and transient charges in the circuit, Fig. 4a.

rent, however, gradually decreases and finally it disappears altogether. The physical reason for this phenomenon is that when the voltage is first impressed on the circuit the capacitor C has no charge at all and acts as though its plates were short-circuited. Hence, for that instant, the circuit may be considered as consisting only of the resistance R . As time proceeds charge accumulates on the condenser until it becomes fully charged and has a voltage between its plates equal and opposite to the impressed voltage E . Under these conditions no current can flow through the circuit because the impressed voltage E is fully neutralized by the potential difference across the condenser plates.

(2) *Continuous Emf. (Short-Circuit).* — Having fully charged the condenser, Fig. 4a, let the switch S_2 be closed and the switch S_1 be simultaneously opened, thereby short-circuiting the condenser through the resistance R . Since the capacitor has a potential difference equal and opposite to the formerly applied voltage E , a current i_2 flows through

the short-circuit. This current has the same shape as i_1 but an opposite direction. Moreover its initial value is equal to the initial value of the charging current, and it virtually vanishes in a time t_1 equal to the time it took the current i_1 to become zero.

(3) *The Time Constant of the Circuit.* — Consider two series circuits with capacitances C_1 and $C_2 > C_1$. Let the resistances of both circuits be identical and let the experiments described above be performed on each of the circuits. The curves which would be obtained are shown in Fig. 5. Note that i_2 (corresponding to the circuit having a capacitance C_2) takes a longer time to become zero. We thus conclude that for a given resistance the greater the capacitance the longer is the time required to attain steady state conditions. We shall show later that

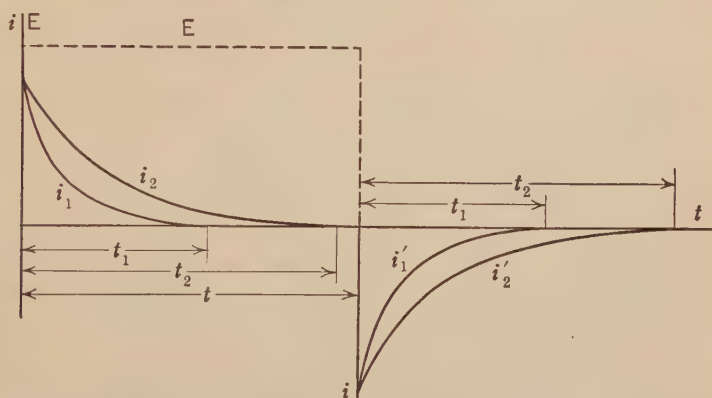


FIG. 5. Effect of the time constant on the transient current in the circuit, Fig. 4a.

the rate at which the transient current vanishes is a function of the product RC . This product is generally known as the *time constant* of the circuit. Its physical dimension (see Chapter IV) is time.

(4) *Alternating Sine Emf. (Establishing the Circuit).* — Let a sine Emf. e be impressed on the series circuit, Fig. 6a, while the switch S_2 is open. The shape of the current depends altogether upon the instant when the switch is closed. If the switch is closed at such an instant $t = 0$ that the current would normally be at its maximum then no transient occurs because at that instant the plates of the condenser are devoid of any charge, and may be considered as if they were short-circuited (see paragraph 1). Consequently there will be a rush of current through the resistance which offers no electrical inertia whatever.

On the other hand, should S_1 be closed at any other instant $t = 0$ then a transient current i_1 (Fig. 6c) appears which distorts the sine current i .

The current element of an oscillograph registers the algebraic sum i' (Fig. 6c) of the two currents i and i_1 .

(5) *Alternating Sine Emf. (Short-Circuit)*. — Assume that steady conditions have been established in the circuit, Fig. 6a, and let S_2 be closed and S_1 be opened at the instant $t = 0$. The value of the transient current that would be obtained depends altogether on the instant when

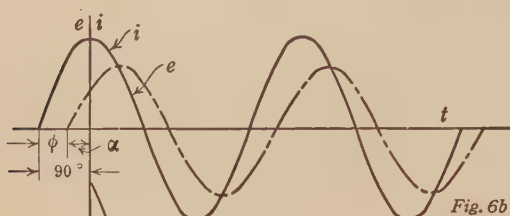


Fig. 6b

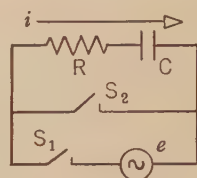


Fig. 6a

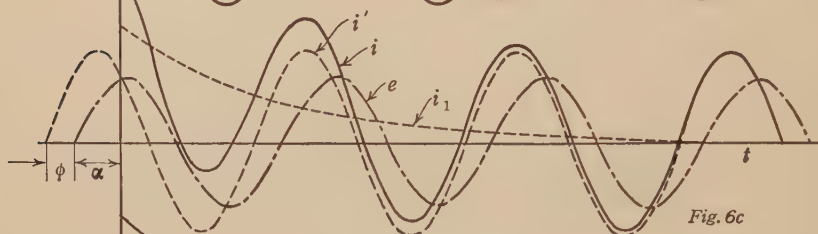


Fig. 6c

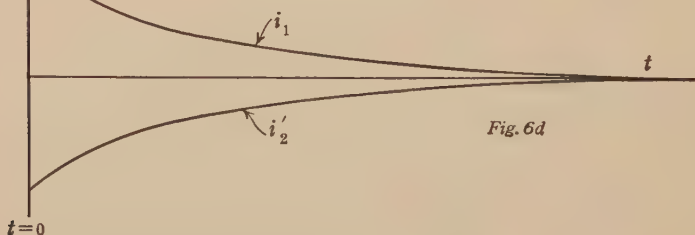


Fig. 6d

FIG. 6a. A series circuit with R and C and a sine Emf. e .

FIG. 6b. Emf. and current waves for the steady state in the circuit, Fig. 6a.

FIG. 6c. Emf. and current waves for the transient state when the Emf. is suddenly impressed on the circuit, Fig. 6a.

FIG. 6d. Current waves when the circuit, Fig. 6a, is suddenly short-circuited.

the switching operation takes place. Should the switching be done at such a time that the potential drop across the condenser is zero then there would be no transient current at all. If, however, the switching occurs at any other instant, then a transient current, having the shape of the curves i_2 , Fig. 6d, would be obtained. The initial value of the transient current is a function of the resistance of the circuit and the charge existing on the condenser plates at the time of short-circuit.

(6) *Alternating Non-Sine Emf. (Establishing the Circuit and Short-Circuit).* — It will not be necessary to describe the transient phenomena which occur when the series circuit, Fig. 6a, is subjected to a non-sine Emf. Suffice it to say that each harmonic behaves as though it were acting alone. Therefore, what has been stated in paragraphs (4) and (5) applies to each harmonic taken separately. Moreover, the total current obtained is the sum of the steady and the transient currents due to each harmonic.

B. Mathematical Theory.

The voltage e impressed on the circuits, Fig. 4a or 6a, is, according to Kirchhoff's Emf. law, equal to the sum of the drops across the resistance and capacitance, that is:

$$e = Ri + q/C \quad (24)$$

But:

$$i = dq/dt \quad (25)$$

Substituting Eq. (25) in (24) and dividing throughout by R we have:

$$dq/dt + (1/RC)q = e/R \quad (26)$$

Eq. (26) is another linear differential equation of the first order and first degree with constant coefficients. Its solution is therefore given by Eq. (5.2), Appendix I. The notations used in Eq. (5.1), Appendix I, bear the following correspondence to those in Eq. (26):

$$x \rightsquigarrow q; \quad a \rightsquigarrow 1/RC; \quad b \rightsquigarrow 1/R; \quad y \rightsquigarrow e; \quad t \rightsquigarrow t$$

Making these substitutions in Eq. (5.2), Appendix I, we obtain the following general solution of Eq. (26):

$$q = e^{-t/RC} \int (e/R) e^{t/RC} dt + K e^{-t/RC} \quad (27)$$

This general solution will now be applied to the special cases of establishing the circuit and short-circuit. The results so obtained will then be used to interpret the curves, Figs. 4 to 6.

(1) *Continuous Emf. (Establishing the Circuit).* — Let the continuous Emf. applied to the circuit, Fig. 4a, be E . Substituting this value in Eq. (27) and integrating, we have:

$$q = CE + K e^{-t/RC} \quad (28)$$

The constant K in Eq. (28) may be determined from the physical conditions which exist at the instant $t = 0$. Thus when $t = 0$, $q = 0$. Substituting these values in Eq. (28) we have:

$$0 = CE + K \quad (29)$$

The result of substituting this value of K in Eq. (28) is:

$$\begin{aligned} q &= CE(1 - e^{-t/RC}) & (a) \\ i &= dq/dt = (E/R)e^{-t/RC} & (b) \end{aligned} \quad (30)$$

whence

The charge curve q_1 (Fig. 4b) represents graphically Eq. (30a). It has an initial value $q = 0$ at the instant $t = 0$ and increases exponentially with time until the condenser becomes fully charged. Theoretically it takes an infinite time for the charge to reach its steady value $Q = CE$. Practically however this value is reached within a limited time t_1 .

Eq. (30b) is represented by the current curve i_1 (Fig. 4b). This curve has an initial value $I_0 = E/R$ at the time $t = 0$ and decreases exponentially with time, becoming practically equal to zero within a short interval of time t_1 .

(2) *Continuous Emf. (Short-Circuit)*. — Let the condenser, Fig. 4a, be fully charged to a value $Q = CE_c$. Let S_2 be closed and S_1 be simultaneously opened. This means that at the instant $t = 0$, $E = 0$ and $q = Q$. Substituting these values in Eq. (28) we have:

$$Q = 0 + K \quad (31)$$

Replacing K in Eq. (28) by this value and remembering that for all instants, after short-circuit, $E = 0$ we have:

$$\begin{aligned} q &= Qe^{-t/RC} = CE_c e^{-t/RC} & (a) \\ i &= dq/dt = -(E_c/R)e^{-t/RC} & (b) \end{aligned} \quad (32)$$

whence:

It must be remembered that E_c in Eqs. (32) is not the impressed voltage but the voltage across the condenser plates which is equal to the impressed voltage. The charge and current given by Eq. (32) are represented graphically by the curves q_2 and i_2 , Fig. 4b. They are both exponential curves similar in character to curves q_1 and i_1 .

Now the time constant RC was introduced and defined in paragraph 3a. It was also stated that the rate at which the current or charge changed is a function of the product RC . That such is the case may be easily shown by taking an interval of time $t = RC$ and substituting this value in Eq. (30a) thereby obtaining:

$$q = (CE)[(\epsilon - 1)/\epsilon] = .632 Q \quad (33)$$

We may then define the *time constant* of a capacitive circuit as such an interval of time $t = RC$, measured from the instant $t = 0$, that at the end of it, the charge shall have reached $(\epsilon - 1)/\epsilon$ or approximately two-thirds of its final value. Evidently then the greater the product RC the longer it takes the charge to attain two-thirds of its final value and hence the longer the total time during which the transient persists.

(3) *Alternating Sine Emf. (Establishing the Circuit).* — Let the applied voltage e in Eq. (24) be a sine function of time. Let its instantaneous value be expressed by Eq. (12). Substituting Eq. (12) in Eq. (27) we have:

$$q = \epsilon^{-t/RC} \int (E_m/R) \sin(\omega t + \alpha) \epsilon^{t/RC} dt + K \epsilon^{-t/RC} \quad (34)$$

Following the procedure outlined in Eqs. (15) to (19) we obtain

$$q = -(E_m/Z_c \omega) \cos(\omega t + \alpha + \phi) + K \epsilon^{-t/RC} \quad (35)$$

In order to apply Eqs. (35) to the case of establishing the circuit with sine Emfs, it must be recalled that at the time when the switch S_1 (Fig. 6a) is closed $q = 0$ and $t = 0$. Substituting these values in Eq. (35) we have:

$$0 = -(E_m/Z_c \omega) \cos(\alpha + \phi) + K \quad (36)$$

Substituting this value of K in Eq. (35) we obtain:

$$q = -(E_m/Z_c \omega) [\cos(\omega t + \alpha + \phi) - \cos(\alpha + \phi) \epsilon^{-t/RC}] \quad (37)$$

Differentiating Eq. (37) we have:

$$\begin{aligned} i = dq/dt &= (E_m/Z_c) [\sin(\omega t + \alpha + \phi) - (1/R\omega C) \cos(\alpha + \phi) \epsilon^{-t/RC}] \quad (a) \\ &= I_m [\sin(\omega t + \alpha + \phi) - (X_c/R) \cos(\alpha + \phi) \epsilon^{-t/RC}] \quad (b) \end{aligned} \quad (38)$$

Eq. (38) gives the value of the current i flowing through the circuit, Fig. 6a. This current is plotted in Fig. 6b for the case when the transient term is zero, that is, when $\cos(\alpha + \phi) = 0$, and in Fig. 6c for the case when the transient term is not zero, that is, when $\cos(\alpha + \phi) \neq 0$.

Now $I_m X_c$ (see Eq. 38b) represents the amplitude E_{mc} of the drop across the condenser and $-I_m X_c \cos(\alpha + \phi) = E_{mc} \sin(\alpha + \phi - 90^\circ)$ represents the instantaneous value of the drop e'_c across the condenser which would normally exist, under steady state, at the instant $t = 0$. Physically, therefore, the transient term in Eq. (38) represents a current whose initial value is equal to e'_c/R . This transient current, in practice, vanishes within a few cycles (or even a fraction of a cycle) after establishing the circuit.

(4) *Sine Emf. (Short-Circuit).* — Assume steady conditions to have been established and let the switch S_2 (Fig. 6a) be closed while S_1 is simultaneously opened. If time is counted from the instant of closing S_2 then when $t = 0$, $E = 0$ and $q = Q_0 = -(E_m/Z_c \omega) \cos(\alpha + \phi)$. Substituting these values in Eq. (35) we have:

$$-(E_m/Z_c \omega) \cos(\alpha + \phi) = K \quad (39)$$

Substituting this value of K in Eq. (35) and remembering that for all instants after short-circuit the steady term is zero we have:

$$q = -(E_m/Z_c\omega) \cos(\alpha + \phi)e^{-t/RC} \quad (40)$$

which, upon differentiation gives:

$$\begin{aligned} i = dq/dt &= (E_m/Z_cR\omega C) \cos(\alpha + \phi)e^{-t/RC} \quad (a) \\ &= (I_m X_c/R) \cos(\alpha + \phi)e^{-t/RC} \quad (b) \end{aligned} \quad (41)$$

The term $I_m X_c \cos(\alpha + \phi)$ in Eq. (41b) represents the potential drop across the capacitor at the instant of short-circuit. Eq. (41) therefore states that the transient current obtained through short-circuit has an initial value equal to the potential drop across the capacitance (at the instant of short-circuit) divided by the resistance, and that the current decreases exponentially with time and vanishes completely when $t = \infty$. In practice this transient current persists only for a short time, after which it either disappears altogether or becomes hardly measurable.

The short-circuit current represented by Eq. (41) is shown in Fig. 6d for the following three cases:

- (a) Short-circuit occurs at a time when the instantaneous value of the Emf. e'_c across the condenser is positive (see curve i_1).
- (b) Short-circuit occurs at a time when e'_c is zero (see the time axis t).
- (c) Short-circuit occurs at a time when e'_c is negative (see curve i'_2).

(5) *Non-Sine Emf. (Establishing the Circuit and Short-Circuit).* — A detailed development of the mathematical theory for non-sine waves becomes superfluous when it is recalled that each harmonic behaves as though it were acting alone. Hence what holds true of sine Emfs. applies equally well to each harmonic. Thus extending the results to the case of non-sine Emfs. we have:

General Solution:

$$q = \sum -(E_{mn}/Z_{cn}\omega_n) \cos n(\omega_1 t + \alpha_n + \phi_n/n) + \sum K_n e^{-t/RC} \quad (35')$$

Case of Establishing the Circuit:

$$q = \sum -(E_{mn}/Z_{cn}\omega_n) [\cos n(\omega_1 t + \alpha_n + \phi_n/n) - \cos(n\alpha_n + \phi_n)e^{-t/RC}] \quad (37)$$

$$i = \sum (E_{mn}/Z_{cn}) [\sin n(\omega_1 t + \alpha_n + \phi_n/n) - (1/R\omega_n C) \cos(n\alpha_n + \phi_n)e^{-t/RC}] \quad (38)$$

Case of Short-Circuit:

$$q = \sum -(E_{mn}/Z_{cn}\omega_n) \cos(n\alpha_n + \phi_n)e^{-t/RC} \quad (39)$$

$$i = \sum (I_{mn} X_{cn}/R) \cos(n\alpha_n + \phi_n)e^{-t/RC} \quad (40)$$

PROBLEMS

1. Cite instances in electrical engineering practice where transient currents do occur.

2a. An induction coil has an inductance $\mathcal{L} = 0.12$ mh. and a resistance $R = 5$ ohms. This coil is connected to a continuous Emf. source. What resistance must be inserted in series with the coil so that the current shall rise to one tenth of its initial value within 0.01 sec. after establishing the circuit?

2b. A 15-ohm resistance possessing an inductance $\mathcal{L} = 0.25$ mh. is connected to a 220-volt, D.C. bus. How soon after the instant of establishing the circuit will the current rise to 5 amp.?

2c. The current in an induction coil must drop to one tenth of its initial value in 0.01 sec. after short-circuit occurs. What must be the time constant of the coil?

2d. The principle of operation of a certain type of voltage regulator is illustrated by the diagram, Fig. 7a. E is the generator voltage, R and $\mathcal{L}$ are the resistance and inductance of the field and R' is an external resistance connected in series with the field. S is a switch connected across R' and is automatically operated by the regu-

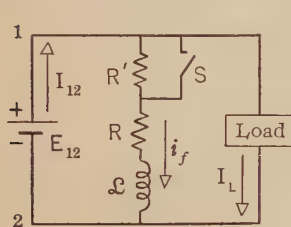


Fig. 7a

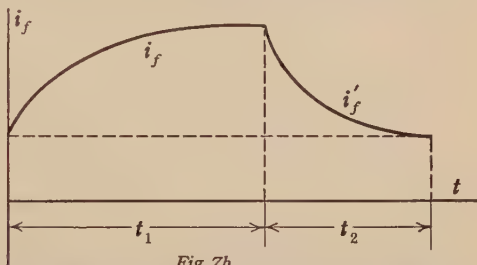


Fig. 7b

FIG. 7a. A circuit illustrating the principle of operation of a voltage regulator.

FIG. 7b. The field currents when S is closed and opened.

lator mechanism. When S is open the field current i_f is decreased and hence the Emf. of the machine is also decreased. When S is closed R' is shunted and the field current i_f increases thereby increasing the voltage E of the machine. It is experimentally observed that the response of the generator to voltage regulation is more sluggish when the Emf. is below normal than when it is above normal. Oscillograms of the field current give the curves i_f and i'_f .

(a) Which of these curves is for S closed?

(b) Explain why $t_1 > t_2$.

(c) How does the fact that $t_1 > t_2$ account for the phenomenon observed regarding the response of the machine to voltage regulation?

2e. An induction coil having a resistance $R = 7$ ohms and an inductance $\mathcal{L} = 0.2$ mh. is subjected to an alternating sine Emf. having an effective value $E = 110$ volts. Oscillograms show that the circuit was established when the voltage had an instantaneous value $e = 50$ volts, and was increasing.

(a) Determine the instantaneous value of the steady current at the time of establishing the circuit, assuming the frequency to be 60 cycles per second.

(b) Draw the curves of the transient, the steady, and the actual current flowing through the circuit.

2f. At which two instantaneous values of the voltage may the circuit in problem 2e be established in order that the transient current be zero?

2g. A series circuit consisting of a resistance $R = 5$ ohms and an inductance of $\mathcal{L}$ mh. was short-circuited. An oscillogram showed that the current at the instant of short-circuit was 20 amp. and that 0.001 sec. after short-circuit the current was 1 amp. Determine the inductance $\mathcal{L}$ of the circuit.

2h. After steady conditions have been established in the circuit, Fig. 8, the switch S is closed. Assuming that the steady current initially flowing through the circuit is I_1 and that the final steady current after S has been closed is I_2 , set up the expression for the transient state in terms of I_1 , I_2 , and the characteristics of the circuit.

2i. Let the circuit, Fig. 8, be subjected to a sine voltage having an amplitude

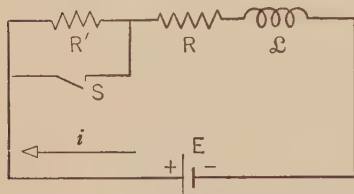


FIG. 8. A series circuit consisting of R , $\mathcal{L}$, and E .

$E = 150$ volts and a frequency of 60 cycles per second. Let the characteristics of the circuit be $R_1 = 10$ ohms, $R_2 = 6$ ohms and $\mathcal{L} = 0.8$ mh. Determine the instantaneous value of the current 0.01 sec. after S has been closed, if the switch is closed at such a time $t = 0$ that the instantaneous value of the impressed voltage is $e_0 = 130$ volts and is increasing.

2j. At what two instantaneous values of the impressed voltage may the switch S be closed in order that the transient current (problem 2i) shall be zero?

2k. Two induction coils having resistances $R_1 = 3$ ohms, $R_2 = 5$ ohms, and inductances $\mathcal{L}_1 = 0.4$ mh., $\mathcal{L}_2 = 0.6$ mh. are connected in series across a 110-volt continuous Emf. source. After steady conditions have been established, a switch is closed thereby short-circuiting the first coil. Determine the expressions for:

- The current in the short-circuited coil.
- The current in the second coil.
- The current in the short-circuiting switch which carries both currents.
- Compute the values of the three currents specified in parts (a), (b) and (c), 0.001 sec. after short-circuit occurs.

2l. Assume that a 60-cycle sine voltage having an amplitude of 300 volts is impressed on the coils specified in problem 2k. Consider steady conditions to have been established and the short-circuit to have occurred at such an instant that the voltage is at its maximum.

- Compute, for the instant $t = 0.001$ sec., the values of the currents flowing in:
 - The short-circuited coil.
 - The second-coil.
 - The switch.

- Plot the curves of currents for each part of the circuit.

2m. Let the circuit specified in problem 2i be subjected to a non-sine wave consisting of a fundamental and a 10 per cent third harmonic. Let the amplitude and frequency of the fundamental be $E_{m1} = 150$ volts and $f_1 = 60$ cycles per second. Let the phase angle between adjacent zero values of the fundamental and third harmonic be 20° measured to the scale of the fundamental. Assuming that the

switch S , Fig. 8, is closed at such an instant $t = 0$ that the fundamental wave has an instantaneous value $e_0 = 40$ volts and is decreasing.

- (a) Determine at the instant of closing the switch:
 - (1) The current due to the fundamental.
 - (2) The current due to the third harmonic.
 - (3) The current due to both.
- (b) Plot the sine waves of the steady and the transient currents due to both the fundamental and third harmonic.
- (c) Draw the record of the current that would be registered by an oscillograph connected in the circuit.

Note. — Neglect the skin effect.

2n. A 60-cycle turbo-alternator supplies an 80 per cent P.f. lagging load. Then suddenly the turbine drops its speed reducing the frequency to 55 cycles. Oscillograms show that such a drop in speed occurred at a time when the voltage in the first phase was at its positive maximum. Assuming that the resistance of the load is 50 ohms per phase, and that the regulator kept the voltage constant at its initial value of 2200 volts.

- (a) Determine the instantaneous value expression for the steady current that would flow at the new frequency.
- (b) Compute the value of the transient current at the instant $t = 0$.
- (c) Plot the current curves obtained in parts *a* and *b*.
- (d) Add the two curves plotted in part *c*. This gives the current oscillogram for the first phase.

2o. Show that in a resistance circuit devoid of inductance there exists no transient current.

2p. A continuous Emf. is impressed on a series circuit having a resistance R and an inductance $\mathcal{L}$. This Emf. is varied with time so that at any instant t its value is $E_t = E_0(1 + at)$.

- (a) Derive from the general solution the expression for the current i .
- (b) Give the expression for the value of the constant of integration for the condition of establishing the circuit.

3a. A series circuit consists of a resistance $R = 10$ ohms and a capacitance $C = 0.8 \mu f$. Plot the curves for i and q when a continuous voltage $E = 110$ volts is impressed on the circuit.

3b. Assuming that the circuit (problem 3a) has been subjected to the 110-volt continuous Emf. for a long enough time to fully charge the condenser. Plot the q and i curves for short-circuit.

3c. Compute the time at which the transient current in problem 3a assumes a value equal to 0.01 of its initial value.

3d. What is the time constant of the circuit specified in problem 3a?

3e. The circuit, problem 3a, is subjected to a sine Emf. having an amplitude $E_m = 15$ volts and a frequency $f = 50,000$ cycles per second. Moreover, at the time of establishing the circuit the instantaneous value of the voltage wave is $e_0 = 10$ volts and its slope is positive.

- (a) Determine the steady current.
- (b) Compute the transient current at the instant $t = 0$.
- (c) Plot the current curve that would be registered by an oscillograph connected in the circuit.

3f. Let the circuit, problem 3e, be short-circuited at an instant when the value of the voltage wave is a positive maximum.

- (a) Compute the transient current at the time when short circuit occurs.
- (b) Plot the curve of the transient current.

3g. The circuit, problem 3a, is subjected to a non-sine wave consisting of a fundamental and a 5 per cent seventh harmonic. The amplitude and frequency of the fundamental are respectively, $E_{m1} = 14$ volts and $f_1 = 70,000$ cycles per second. Moreover, the phase angle between two adjacent zero values of the fundamental and seventh harmonic is zero. Assuming the circuit to be established at a time when the instantaneous value of the fundamental is a positive maximum,

- (a) Determine the analytical expressions for the steady currents due to the fundamental and seventh harmonic respectively.
- (b) Compute the values of the transient currents due to the fundamental and seventh harmonic at the instant of establishing the circuit.
- (c) Plot the currents obtained in parts a and b.
- (d) Add the four curves thereby obtaining the current oscillogram.

CHAPTER XVIII

ELECTRIC TRANSIENTS

PART B — DOUBLE-ENERGY TRANSIENTS

1. Introduction. — Although, for purposes of analysis circuits may be divided into those consisting of resistance and inductance as well as those consisting of resistance and capacitance, yet, actually, no circuit is completely devoid of any of these characteristics. It is therefore essential to treat the general case of a circuit consisting of resistance, inductance and capacitance.

The transient state in such a circuit presents some very interesting problems. It will be recalled that the electromagnetic energy of the circuit is stored in the inductance and the electrostatic energy is stored in the capacitance. Moreover, Fig. 4c, page 155, shows that during the interval when energy is stored in the inductance (see positive part of the p_L -curve) energy is being delivered from the capacitor (see negative parts of the p_C -curve) and vice versa. Consequently there exists an interchange of energy between the Emf. source on the one hand and $\mathcal{L}$ and C on the other hand such that at the time when the source is storing energy, in $\mathcal{L}$, it is receiving energy from C . This amounts to a periodic interchange of energy between $\mathcal{L}$ and C . It may result in free electric oscillations if the proper $\mathcal{L}$ and C are selected. The physical phenomena may be best understood by a study of the results obtained after the differential equation for such a circuit has been solved. We shall therefore proceed to the mathematical treatment and depend upon the results to enlighten us as to the physical conditions which prevail in such circuits.

2. The General Solution of the Differential Equation for a Series Circuit Consisting of Resistance, Inductance, and Capacitance. — According to Kirchhoff's Emf. law the voltage across a series circuit consisting of R , $\mathcal{L}$, and C is:

$$e = Ri + \mathcal{L}(di/dt) + (q/C) \quad (1)$$

Replacing i in Eq. (1) by dq/dt and di/dt by d^2q/dt^2 , we have:

$$d^2q/dt^2 + (R/\mathcal{L})(dq/dt) + (1/\mathcal{L}C)q = (1/\mathcal{L})e \quad (2)$$

Eq. (2) is a linear differential equation of the second order with constant coefficients. It corresponds to Eq. (6.1), Appendix I, and its general solution is given by Eq. (6.2).

Comparing Eq. (2) with Eq. (6.1), Appendix I, we find that:

$$\begin{array}{ccccccc}
 q \text{ in Eq. (2)} & \text{corresponds to} & x \text{ in Eq. (6.1)} \\
 t & " & " & " & " & " & t & " & " & " \\
 R/\mathcal{L} & " & " & " & " & " & a & " & " & " \\
 1/\mathcal{L}C & " & " & " & " & " & b & " & " & " \\
 1/\mathcal{L} & " & " & " & " & " & h & " & " & " \\
 e & " & " & " & " & " & y & " & " & "
 \end{array}$$

Therefore, the general solution of Eq. (2) is:

$$q = \epsilon^{-vt} \int \left[\epsilon^{-ut} \int (e/\mathcal{L}) \epsilon^{ut} dt + K_1 \epsilon^{-ut} \right] \epsilon^{vt} dt + K_2 \epsilon^{-vt} \quad (3)$$

where (see Eq. 6.7) Appendix I:

$$\left. \begin{aligned}
 u &= (RC + \sqrt{R^2C^2 - 4\mathcal{L}C})/2\mathcal{L}C & (a) \\
 v &= (RC - \sqrt{R^2C^2 - 4\mathcal{L}C})/2\mathcal{L}C & (b) \\
 uv &= 1/\mathcal{L}C & (c) \\
 \text{and } u - v &= \sqrt{(R/2\mathcal{L})^2 - (1/\mathcal{L}C)} & (d) \\
 u + v &= R/\mathcal{L} & (e)
 \end{aligned} \right\} \quad (4)$$

3. The Transient State in a Series Circuit, Consisting of Resistance, Inductance, Capacitance, and a Continuous Emf. — Consider the circuit, Fig. 1a. Let the switch S be closed. Let it be required to find the value of the current that would flow in the circuit at any instant t . To do this substitute E for e in Eq. (3) and integrate, thereby obtaining:

$$q = (E/\mathcal{L}uv) + [K_1/(v - u)]\epsilon^{-ut} + K_2\epsilon^{-vt} \quad (5)$$

Substituting for the term $(\mathcal{L}uv)$ in Eq. (5) its value from Eq. (4c) and using K'_1 for $K_1/(v - u)$ we have:

$$q = CE + K'_1\epsilon^{-ut} + K_2\epsilon^{-vt} \quad (6)$$

Differentiating Eq. (6) with respect to t we have:

$$i = dq/dt = -(K'_1u)\epsilon^{-ut} - (K_2v)\epsilon^{-vt} \quad (7)$$

Eqs. (6) and (7) give only the shape of the charge and current curves. They do not, however, give the definite values of q and i at any time t . Before these values can be evaluated it will be necessary to determine the constants K'_1 and K_2 . To do this we shall examine the physical conditions prevailing at the time of establishing the circuit. These are:

- (a) The condenser is assumed to be uncharged. Hence $q = 0$ at $t = 0$.
- (b) Owing to the inductance or inertia of the circuit the current is zero or $i = 0$ at $t = 0$.
- (c) The Emf. E , is suddenly impressed when S is closed or $E = E$ at $t = 0$.

Substituting these values in Eqs. (6) and (7) we obtain:

$$\begin{cases} 0 = CE + K'_1 + K_2 & (a) \\ 0 = K'_1 u + K_2 v & (b) \end{cases} \quad (8)$$

Solving Eqs. (8) simultaneously for K'_1 and K_2 we have:

$$\begin{cases} K'_1 = +CEv/(u - v) & (a) \\ K_2 = -CEu/(u - v) & (b) \end{cases} \quad (9)$$

Substituting Eqs. (9) in (6) and (7) we have:

$$\begin{cases} q = CE - \{CE/(u - v)\} \{u\epsilon^{-vt} - v\epsilon^{-ut}\} & (a) \\ i = (dq/dt) = \{CEuv/(u - v)\} \{\epsilon^{-vt} - \epsilon^{-ut}\} & (b) \end{cases} \quad (10)$$

Now the values of u and v in Eqs. (10) may be either real or complex depending upon the relative magnitudes of R^2C^2 and $4\mathcal{L}C$. If (see Eqs. 4) $R^2C > 4\mathcal{L}$, then u and v are real. If $R^2C < 4\mathcal{L}$, then u and v are complex. Hence it will be necessary to consider the following three cases:

- A. The Dead-Beat Case ($R^2C > 4\mathcal{L}$)
- B. The Oscillatory Case ($R^2C < 4\mathcal{L}$)
- C. The Limiting Case ($R^2C = 4\mathcal{L}$)

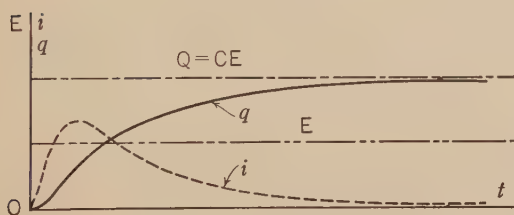


Fig. 1b

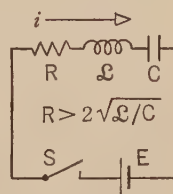


Fig. 1a

FIG. 1a. A series circuit consisting of E , R , $\mathcal{L}$, and C such that $R > 2\sqrt{\mathcal{L}/C}$ (Dead-beat case).

FIG. 1b. Curves of the transient current and transient charge in the circuit, Fig. 1a.

Case A (the Dead-Beat Case) $R^2C > 4\mathcal{L}$ or $R > 2\sqrt{\mathcal{L}/C}$. — If the characteristics of the circuit (Fig. 1a) are such that $R > 2\sqrt{\mathcal{L}/C}$, then

u and v in Eqs. (10) are real. The charge q increases exponentially with time until the condenser is fully charged to a value $Q = CE$. The curve q has a point of inflection at time t_1 as shown in Fig. 1b. The current i , on the other hand, increases exponentially with time up to t_1 , after which it decreases to zero. Fig. 1b shows that when the charge has attained its maximum value Q the current is zero. Physically this means that the potential drop across the capacitance is exactly equal and opposite to the applied Emf. E . Hence no current can flow through the circuit because the applied Emf. is completely neutralized by the drop across the condenser.

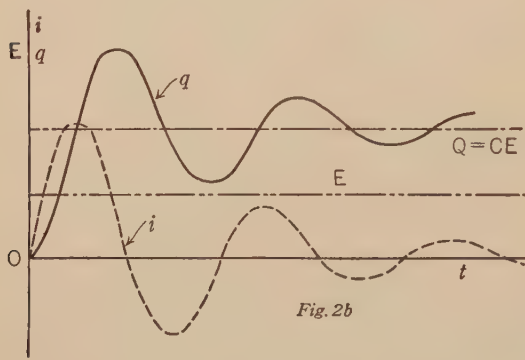


Fig. 2b

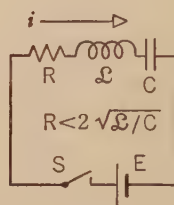


Fig. 2a

FIG. 2a. A series circuit consisting of E , R , L , and C such that $R < 2\sqrt{L/C}$ (Oscillatory case).

FIG. 2b. Curves of the transient current and transient charge in the circuit, Fig. 2a.

Case B (the Oscillatory Case) $R < 2\sqrt{L/C}$. — If the characteristics of the circuit, Fig. 2a, are such that $R < 2\sqrt{L/C}$, then u and v in Eqs. (10) are both complex because the quantity under the radical in Eq. (4) is imaginary. Let:

$$\beta = (\sqrt{R^2C^2 - 4LC})/2LC \quad (11)$$

Then Eqs. (4) become:

$$\left. \begin{aligned} u &= (R/2L) + j\beta & (a) \\ v &= (R/2L) - j\beta & (b) \\ w &= 1/LC & (c) \\ u - v &= 2j\beta & (d) \end{aligned} \right\} \quad (12)$$

(1) *The Oscillatory Charge.* — Substituting Eqs. (12) in Eq. (10a) and rearranging terms we obtain:

$$\begin{aligned}
 q &= CE \left[1 - \epsilon^{-Rt/2\mathcal{L}} \left\{ \frac{R}{2\mathcal{L}\beta} \frac{\epsilon^{j\beta t} - \epsilon^{-j\beta t}}{2j} - \frac{\epsilon^{j\beta t} + \epsilon^{-j\beta t}}{2} \right\} \right] \quad (a) \\
 &= CE \left[1 - \epsilon^{-Rt/2\mathcal{L}} \left\{ (R/2\mathcal{L}\beta) \sin \beta t - \cos \beta t \right\} \right] \quad (b) \\
 &= CE \left[1 - \epsilon^{-Rt/2\mathcal{L}} (\sqrt{(R/2\mathcal{L}\beta)^2 + 1}) \sin (\beta t - \phi) \right] \quad (c)
 \end{aligned} \tag{13}$$

where

$$\phi = \tan^{-1} (2\mathcal{L}\beta/R) = \tan^{-1} \sqrt{1 - (4\mathcal{L}/R^2C)} \tag{14}$$

Substituting the value of β from Eq. (11) in Eq. (13c) we have:

$$\begin{aligned}
 q &= CE \left[1 - \left\{ \epsilon^{-Rt/2\mathcal{L}} \sqrt{\frac{2R^2C - 4\mathcal{L}}{R^2C - 4\mathcal{L}}} \right\} \sin (\beta t - \phi) \right] \quad (a) \\
 &= Q - q_m \sin (\beta t - \phi) \quad (b)
 \end{aligned} \tag{15}$$

Eq. (15b) shows that the charge consists of two terms: (a) a constant term $Q = CE$, and (b) a sine term having an angular velocity β defined by Eq. (11), a phase angle ϕ defined by Eq. (14) and a variable amplitude q_m such that:

$$q_m = CE \left\{ \frac{2R^2C - 4\mathcal{L}}{R^2C - 4\mathcal{L}} \right\} \epsilon^{-Rt/2\mathcal{L}} \tag{16}$$

The amplitude q_m decreases exponentially with time so that when t is infinite $q_m = 0$ and the charge on the condenser is the constant charge $Q = CE$.

A sine function whose amplitude decreases with time is known as a *decremental or damped sine function* and the factor $\epsilon^{-Rt/2\mathcal{L}}$, which accounts for the decrease in the amplitude, is known as the decrement or damping factor.

Fig. 2b shows the charge q , Eq. (15b), as well as one of its components the constant charge Q . The other component, $q_m \sin (\beta t - \phi)$, is not shown because its shape is similar to that of q .

(2) *The Oscillatory Current.* — Substituting Eqs. (12) in Eq. (10b) and simplifying we have:

$$i = \frac{E\epsilon^{-Rt/2\mathcal{L}}}{\mathcal{L}\beta} \left[\frac{\epsilon^{j\beta t} - \epsilon^{-j\beta t}}{2j} \right] = \frac{2CE\epsilon^{-Rt/2\mathcal{L}}}{\sqrt{R^2C^2 - 4\mathcal{L}C}} \sin \beta t = i_m \sin \beta t \tag{17}$$

$$\text{where} \quad i_m = [2CE/\sqrt{R^2C^2 - 4\mathcal{L}C}] \epsilon^{-Rt/2\mathcal{L}} \tag{18}$$

Eq. (17) shows that the current is a decremental sine function having an angular velocity β expressed by Eq. (11) and a variable amplitude i_m expressed by Eq. (18). As time t increases i_m decreases until when $t_1 = \infty$ the current vanishes. The current curve is given in Fig. 2b.

The fact that the charge and current are oscillatory shows that the energy supplied to the circuit is stored alternately first in the capacitance and then in the inductance, a part of it being lost as I^2R loss in the

resistance. When steady conditions are finally established, all the stored energy in the circuit is concentrated in the condenser. It is possible that some of the energy is radiated into space. A discussion of this radiated energy falls outside the scope of this text.

Case C (the Limiting Case) $R = 2\sqrt{\mathcal{L}/C}$. — Let the characteristics of the circuit, Fig. 3a, be such that $R = 2\sqrt{\mathcal{L}/C}$. Then $\beta = 0$ and $u = v$. Hence both q and i in Eqs. (10) become infinite. Physically this is known not to be the case. It will be necessary, therefore, to evaluate Eqs. (10) by assuming that β is not zero, but an infinitely small quantity whose value approaches zero. Under these conditions Eq. (6) becomes:

$$\left. \begin{aligned} q &= CE + K'_1 e^{-\{(R/2\mathcal{L})+\beta\}t} + K_2 e^{-\{(R/2\mathcal{L})-\beta\}t} \quad (a) \\ &= CE + e^{-Rt/2\mathcal{L}}(K'_1 e^{-\beta t} + K_2 e^{\beta t}) \quad (b) \end{aligned} \right\} \quad (19)$$

But:

$$\left. \begin{aligned} e^{\beta t} &= 1 + \beta t + (\beta^2 t^2/2!) + (\beta^3 t^3/3!) + \dots \quad (a) \\ e^{-\beta t} &= 1 - \beta t + (\beta^2 t^2/2!) - (\beta^3 t^3/3!) + \dots \quad (b) \end{aligned} \right\} \quad (20)$$

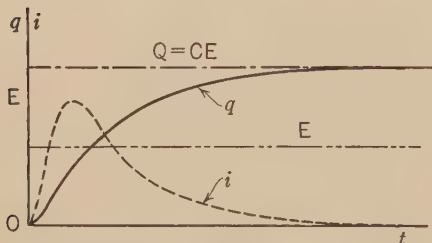


Fig. 3b

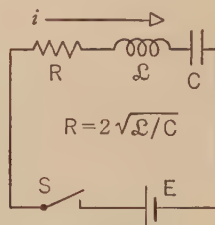


Fig. 3a

FIG. 3a. A series circuit consisting of E , R , $\mathcal{L}$, and C such that $R = 2\sqrt{\mathcal{L}/C}$ (Limiting case).

FIG. 3b. Curves of the transient current and transient charge in the circuit Fig. 3a.

Since, by assumption, β is extremely small, all higher powers of β may be neglected. Hence substituting the first two terms of Eqs. (20) in (19) we have:

$$q = CE + e^{-Rt/2\mathcal{L}}[K'_1(1 - \beta t) + K_2(1 + \beta t)] \quad (21)$$

Expanding the quantities within brackets in Eq. (21) and factoring t we have:

$$q = CE + e^{-Rt/2\mathcal{L}}[A + \beta t] \quad (22)$$

whence by differentiation:

$$\dot{q} = dq/dt = -(R/2\mathcal{L})e^{-Rt/2\mathcal{L}}[A + \beta t] + B e^{-Rt/2\mathcal{L}} \quad (23)$$

In order to determine the constants A and B in Eqs. (22) and (23) we have the conditions that when $t = 0$, $q = 0$ and $\dot{q} = 0$. Substituting

these values in Eqs. (22) and (23) we obtain:

$$\begin{cases} 0 = CE + A & (a) \\ 0 = -A(R/2\mathfrak{L}) + B & (b) \end{cases} \quad (24)$$

Solving Eqs. (24) for A and B and substituting the values so obtained in Eqs. (22) and (23) we have:

$$q = CE \left[1 - e^{-Rt/2\mathfrak{L}} \left\{ 1 + Rt/2\mathfrak{L} \right\} \right] \quad (25)$$

$$i = (CER^2t/4\mathfrak{L}^2) e^{-Rt/2\mathfrak{L}} \quad (26)$$

The curves representing Eqs. (25) and (26) are plotted in Fig. 3b. These curves are not oscillatory. In fact they are very similar to the curves shown in Fig. 1b.

4. The Transient State in a Short-Circuited Series Circuit Consisting of Resistance, Inductance, and Capacitance. — Let the capacitor, Fig. 4a, be completely charged. Then let the switch S_2 be closed and

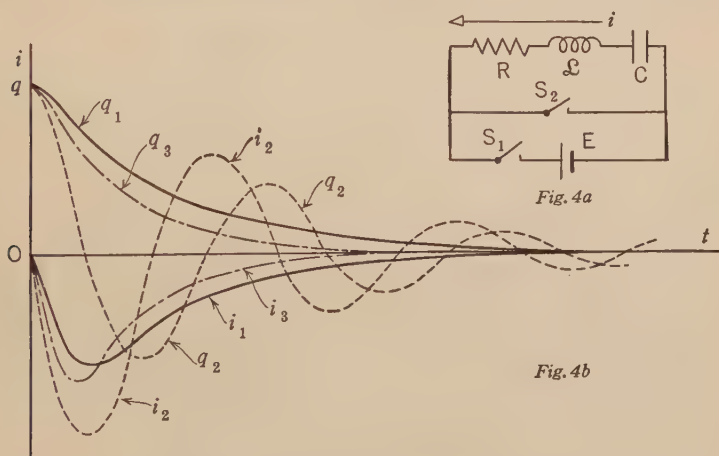


FIG. 4a. A series circuit consisting of E , R , $\mathfrak{L}$, and C .

FIG. 4b. Transient currents and transient charges obtained by short-circuiting the circuit, Fig. 4a.

Dead-Beat Case ($R > 2\sqrt{\mathfrak{L}/C}$) gives curves i_1 and q_1 .

Oscillatory Case ($R < 2\sqrt{\mathfrak{L}/C}$) gives curves i_2 and q_2 .

Limiting Case ($R = 2\sqrt{\mathfrak{L}/C}$) gives curves i_3 and q_3 .

the switch S_1 be simultaneously opened. This establishes a short-circuit. The curves of charge and current (Fig. 4b) may be any two of the curves i_1 and q_1 or i_2 and q_2 or i_3 and q_3 . The problem is to predetermine which of these curves will be obtained for a given circuit and to find the mathematical expression for each of these curves. The

treatment is identical with that given in Art. 3 above. The only difference consists in determining the constants K'_1 and K_2 . These constants are determined from the following physical conditions which prevail at the instant of short circuit:

$$t = 0, \quad E = 0, \quad i = 0, \quad q = Q = CE_c \quad (27)$$

where E_c is the voltage across the condenser plates and is equal to the formerly applied voltage.

Referring to Eqs. (6) and (7) and substituting the values of t , Q , i , and E_c at the instant of short-circuit we have:

$$\left. \begin{aligned} Q &= K'_1 + K_2 & (a) \\ 0 &= K'_1 u + K_2 v & (b) \end{aligned} \right\} \quad (28)$$

$$\text{whence} \quad \left. \begin{aligned} K'_1 &= -Qv/(u - v) & (a) \\ K_2 &= Qu/(u - v) & (b) \end{aligned} \right\} \quad (29)$$

Substituting these values in Eqs. (6) and (7) we obtain:

$$\left. \begin{aligned} q &= [Q/(u - v)] [u\epsilon^{-vt} - v\epsilon^{-ut}] & (a) \\ i &= [Quv/(u - v)] [\epsilon^{-ut} - \epsilon^{-vt}] & (b) \end{aligned} \right\} \quad (30)$$

Comparing Eqs. (30) and (10) we find that the two expressions are identical except for signs and for the absence of the constant term CE in Eq. (30a). Hence it is not necessary to go through the intermediate steps in order to determine the results. The answers may be written directly from the results obtained above by merely omitting the constant term CE and changing the signs of the other terms. We shall give here the results of the three cases:

Case A (Dead-Beat Case) $R > 2\sqrt{\mathcal{L}/C}$. — If the characteristics of the circuit, Fig. 4a, are such that $R > 2\sqrt{\mathcal{L}/C}$ then the charge and current curves are represented by Eqs. (30) where u and v are real numbers. These are curves i_1 and q_1 , Fig. 4b.

Case B (Oscillatory Case) $R < 2\sqrt{\mathcal{L}/C}$. — If the characteristics of the circuit, Fig. 4a, are such that $R < 2\sqrt{\mathcal{L}/C}$ then, since the form of the charge and current equations are similar to Eqs. (10), the solution for q and i is given by equations similar to (15) and (17). Thus omitting the constant term CE and changing signs we have:

$$\left. \begin{aligned} q &= +q_m \sin(\beta t - \phi) & (a) \\ i &= -i_m \sin \beta t & (b) \end{aligned} \right\} \quad (31)$$

where q_m and i_m are expressed by Eqs. (16) and (18) respectively.

Case C (Limiting Case) $R = 2\sqrt{\mathcal{L}/C}$. — The solution for this case is given by equations similar to (25) and (26) respectively. Hence,

omitting the constant term CE and changing signs we obtain:

$$\left. \begin{aligned} q &= Q\epsilon^{-Rt/2L} [1 + (Rt/2L)] & (a) \\ i &= -(QR^2t/4L^2) \epsilon^{-Rt/2L} & (b) \end{aligned} \right\} \quad (32)$$

D. Discussion of the Three Cases. — We have described three cases of transients in a series circuit consisting of R , L , and C . These are:

- (a) The dead-beat case occurring in a circuit whose characteristics are such that $(R > 2\sqrt{L/C})$.
- (b) The oscillatory case occurring in a circuit whose characteristics are such that $(R < 2\sqrt{L/C})$.
- (c) The limiting case occurring in a circuit whose characteristics are such that $(R = 2\sqrt{L/C})$.

Although mathematically, the difference between these three cases is quite evident. Physically the line of demarcation becomes obliterated as the limiting case is approached either from the dead-beat case or from the oscillatory case. We have seen above that the curves of q and i for cases *A* and *C* are very similar. As to case *B*, the oscillations become so rapidly damped as the value of R approaches $2\sqrt{L/C}$ that the case is sometimes, although erroneously, referred to as a dead-beat case.

There exists, however, an intrinsic difference between the three cases which must be here emphasized. The transient state in the limiting case always lasts for a shorter time than in either of the other two cases.*

5. The Transient State in a Series Circuit Consisting of Resistance, Inductance, Capacitance, and a Sine Emf. e (Establishing the Circuit. — Let the switch S_1 in Fig. 5*a* be closed while S_2 is open. Let the switching occur at such an instant $t = 0$ that the instantaneous value of the impressed voltage is:

$$e_0 = E_m \sin \alpha \quad (33)$$

Then the value of the impressed voltage at any time t , after closing S_1 , is:

$$e = E_m \sin (\omega t + \alpha) = (E_m/2j) \left[\epsilon^{j(\omega t + \alpha)} - \epsilon^{-j(\omega t + \alpha)} \right] \quad (34)$$

Substituting this value in Eq. (3) we have:

$$\begin{aligned} q &= \epsilon^{-vt} \int \left[\epsilon^{-ut} (E_m/2j) \int \left\{ \epsilon^{j\alpha} \epsilon^{(u+j\omega)t} - \epsilon^{-j\alpha} \epsilon^{(u-j\omega)t} \right\} dt + K_1 \epsilon^{-ut} \right] \epsilon^{vt} dt \\ &\quad + K_2 \epsilon^{-vt} \end{aligned} \quad (35)$$

* See W. E. Sumpner, Phil. Mag., Ser. 5, 25, p. 453, 1888.

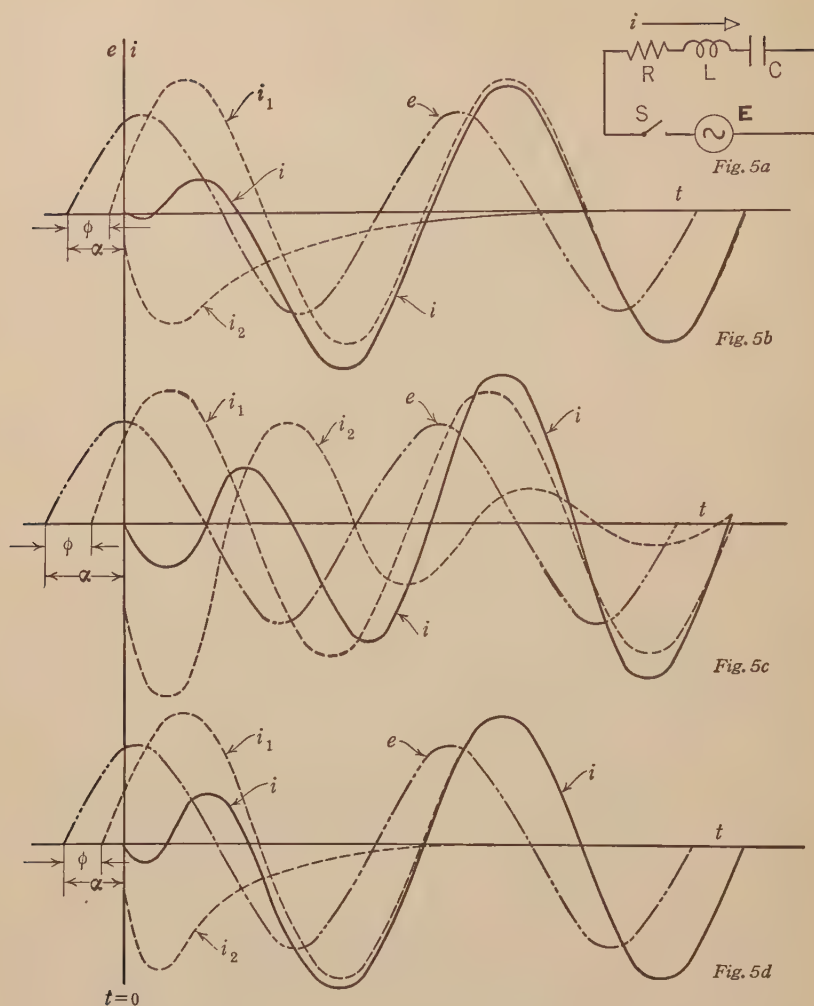


FIG. 5a. A series circuit consisting of R , L , and C and a sine Emf. e .

FIG. 5b. Transient current in the circuit, Fig. 5a, when the characteristics are such that $R > 2\sqrt{L/C}$.

FIG. 5c. Transient current in the circuit, Fig. 5a, when the characteristics are such that $R < 2\sqrt{L/C}$.

FIG. 5d. Transient current in the circuit, Fig. 5a, when the characteristics are such that $R = 2\sqrt{L/C}$.

Integrating and simplifying we obtain:

$$q = \frac{E_m}{2j\mathcal{L}} \left[\frac{\epsilon^{j(\omega t + \alpha)}}{(uv - \omega^2) + j\omega(u + v)} - \frac{\epsilon^{-j(\omega t + \alpha)}}{(uv - \omega^2) - j\omega(u + v)} \right] + K'_1 \epsilon^{-ut} + K_2 \epsilon^{-vt} \quad (36)$$

Substituting in Eq. (36) the values $uv = 1/\mathcal{L}C$ and $u + v = R/\mathcal{L}$ (see Eqs. 4) and simplifying we have:

$$q = -\frac{E_m}{2\omega} \left[\frac{\epsilon^{j(\omega t + \alpha)}}{R + j\{\omega\mathcal{L} - (1/\omega C)\}} + \frac{\epsilon^{-j(\omega t + \alpha)}}{R - j\{\omega\mathcal{L} - (1/\omega C)\}} \right] + K'_1 \epsilon^{-ut} + K_2 \epsilon^{-vt} \quad (37)$$

$$\text{But} \quad R \pm j\{\omega\mathcal{L} - (1/\omega C)\} = Z\epsilon^{\pm j\phi} \quad (38)$$

Hence Eq. (37) may be written in the following form:

$$\left. \begin{aligned} q &= -(E_m/\omega Z) \left[\frac{\epsilon^{j(\omega t + \alpha - \phi)} + \epsilon^{-j(\omega t + \alpha - \phi)}}{2} \right] + K'_1 \epsilon^{-ut} + K_2 \epsilon^{-vt} \quad (a) \\ &= -(E_m/\omega Z) \cos(\omega t + \alpha - \phi) + K'_1 \epsilon^{-ut} + K_2 \epsilon^{-vt} \quad (b) \end{aligned} \right\} (39)^*$$

Differentiating Eq. (39b) we obtain:

$$i = (E_m/Z) \sin(\omega t + \alpha - \phi) - K'_1 u \epsilon^{-ut} - K_2 v \epsilon^{-vt} \quad (40)^*$$

Eq. (40) states that the current, flowing in a series circuit consisting of R , $\mathcal{L}$, and C and a sine Emf. e , comprises a steady term $(E_m/Z) \sin(\omega t + \alpha - \phi)$ and two transient terms $K'_1 u \epsilon^{-ut}$ and $K_2 v \epsilon^{-vt}$ which are essentially the same as the transient terms met with in Art. 4. The two transient terms usually vanish after a few cycles and the only current which flows in the circuit is the steady sine current.

The values of the constants K'_1 and K_2 are determined from the initial conditions. Thus when $t = 0$, $q = 0$ and $i = 0$. Hence Eqs. (39b) and (40) reduce to the form:

$$\left. \begin{aligned} 0 &= (E_m/\omega Z) \cos(\alpha - \phi) + K'_1 + K_2 \quad (a) \\ 0 &= (E_m/Z) \sin(\alpha - \phi) - K'_1 u - K_2 v \quad (b) \end{aligned} \right\} \quad (41)$$

Solving Eqs. (41) for K'_1 and K_2 we obtain:

$$\left. \begin{aligned} K_2 &= \frac{E_m}{Z(v - u)} \left[\sin(\alpha - \phi) + (u/\omega) \cos(\alpha - \phi) \right] \quad (a) \\ K'_1 &= \frac{E_m}{Z(v - u)} \left[\sin(\alpha - \phi) + (v/\omega) \cos(\alpha - \phi) \right] \quad (b) \end{aligned} \right\} \quad (42)$$

where u and v are as given in Eqs. (4)

* In Eqs. (39) and (40) and in all the equations which follow ϕ is positive if $\omega\mathcal{L} > (1/\omega C)$ and negative if $\omega\mathcal{L} < (1/\omega C)$.

Eqs. (39b) and (40) show that the transient terms, in this case, are identical with those obtained in Eqs. (6) and (7) respectively, for the continuous Emf. case. Hence the discussion given in Arts. 3 and 4 applies here with equal force. In other words, one or the other of the three cases of transients may occur depending upon the relative magnitudes of the circuit characteristics. Thus Fig. 5b shows the transient current in a circuit whose characteristics are such that $R > 2\sqrt{\mathcal{L}/C}$. Fig. 5c illustrates the transient state in a circuit whose characteristics are such that $R < 2\sqrt{\mathcal{L}/C}$; and Fig. 5d gives the current curves of a circuit whose characteristics are such that $R = 2\sqrt{\mathcal{L}/C}$.

6. Inevitability of the Transient State in a Series Circuit with Resistance, Inductance, and Capacitance. — It is shown in Chapter XVII, Art. 2, parts 4 and 5, and in Art. 3, parts 4 and 5, that it is possible to time the switching operations in such a way that no transient exists in a series circuit with R and $\mathcal{L}$ or a series circuit with R and C respectively. The object of this article is to show that such is not the case with circuits consisting of R , $\mathcal{L}$, and C .

In order that the transient in a series circuit with R , $\mathcal{L}$, and C shall be zero the values of the constants K'_1 and K_2 must be both identically equal to zero. From Eqs. (42) K'_1 and K_2 are both equal to zero when:

$$\sin(\alpha - \phi) = -(u/\omega) \cos(\alpha - \phi) = -(v/\omega) \cos(\alpha - \phi) \quad (43)$$

This equality may be satisfied if and only if $u = v$ (that is, in the limiting case). Hence in both the dead-beat and the oscillatory cases the transient state does exist and nothing can be done to avoid it.

On the other hand, if $u = v$, then $(u - v) = 0$; and K_2 and K'_1 in Eqs. (42) become infinite. Hence before asserting that the transient state may be avoided in the limiting case it is necessary to evaluate K'_1 and K_2 when $u = v$. To do this we proceed as in Eqs. (19) to (23) and obtain the following expressions for q and i :

$$\left. \begin{aligned} q &= -(E_m/\omega Z) \cos(\omega t + \alpha - \phi) + \epsilon^{-Rt/2\mathcal{L}} [A + Bt] \\ i &= (dq/dt) = (E_m/Z) \sin(\omega t + \alpha - \phi) - (R/2\mathcal{L}) [A + Bt] \epsilon^{-Rt/2\mathcal{L}} \\ &\quad + B \epsilon^{-Rt/2\mathcal{L}} \end{aligned} \right\} \quad (44)$$

When $t = 0$, $i = 0$, $q = 0$; and Eqs. (44) become:

$$\left. \begin{aligned} 0 &= (E_m/\omega Z) \cos(\alpha - \phi) + A & (a) \\ 0 &= (E_m/Z) \sin(\alpha - \phi) - (R/2\mathcal{L})A + B & (b) \end{aligned} \right\} \quad (45)$$

whence

$$\left. \begin{aligned} A &= -(E_m/\omega Z) \cos(\alpha - \phi) & (a) \\ B &= -(E_m/Z) \sin(\alpha - \phi) - (R/2\mathcal{L})(E_m/\omega Z) \cos(\alpha - \phi) & (b) \end{aligned} \right\} \quad (46)$$

Now since α is arbitrary it is possible to make A equal to zero by properly timing the switching operations so that $\cos(\alpha - \phi) = 0$. But if this be done then $\sin(\alpha - \phi) = \pm 1$. Hence $B \neq 0$. Thus the transient state does exist in the limiting case.

We conclude, therefore, that the transient state in a series circuit consisting of R , $\mathcal{L}$, and C and a sine Emf. e can never be avoided.

7. The Transient State in a Series Circuit Consisting of Resistance, Inductance, Capacitance and a Sine Emf. e (Short-Circuit). — Let steady conditions in Fig. 5a be established; then let the source be short-circuited. The magnitude of the transient current and transient charge depends upon the values of q and i at the instant of switching. The shapes of the curves, however, depend upon the ratio of R to $2\sqrt{\mathcal{L}/C}$.

In order to obtain the values of the constants K'_1 and K_2 in Eqs. (39b) and (40), it must be recalled that at the instant $t = 0$, $e = 0$, $q = (E_m/\omega Z) \cos(\alpha - \phi)$ and $i = (E_m/Z) \sin(\alpha - \phi)$. Substituting these values in Eqs. (39b) and (40) we obtain:

$$\left. \begin{aligned} (E_m/\omega Z) \cos(\alpha - \phi) &= K'_1 + K_2 & (a) \\ (E_m/Z) \sin(\alpha - \phi) &= uK'_1 + vK_2 & (b) \end{aligned} \right\} \quad (47)$$

whence:

$$\left. \begin{aligned} K'_1 &= [E_m/Z(v - u)] [(v/\omega) \cos(\alpha - \phi) - \sin(\alpha - \phi)] & (a) \\ K_2 &= [E_m/Z(u - v)] [(u/\omega) \cos(\alpha - \phi) - \sin(\alpha - \phi)] & (b) \end{aligned} \right\} \quad (48)$$

Hence the equations for the short-circuit charge and current are:

$$\left. \begin{aligned} q &= K'_1 \epsilon^{-ut} + K_2 \epsilon^{-vt} & (a) \\ i &= -uK'_1 \epsilon^{-ut} - vK_2 \epsilon^{-vt} & (b) \end{aligned} \right\} \quad (49)$$

where K'_1 and K_2 are expressed by Eqs. (48).

We shall examine the short-circuit transient for the three cases: (a) the Dead-Beat Case, (b) the Oscillatory Case, and (c) the Limiting Case.

A. Dead-Beat Case, $R > 2\sqrt{\mathcal{L}/C}$. — If the characteristics of the circuit are such that $R > 2\sqrt{\mathcal{L}/C}$, then u and v in Eq. (48) are both real and the curves representing q and i are exponential in form. Moreover, the initial value of the charge and current depends upon the values of $\cos(\alpha - \phi)$ and $\sin(\alpha - \phi)$, that is, upon the instant of switching.

B. Oscillatory Case, $R < 2\sqrt{\mathcal{L}/C}$. — If the characteristics of the circuit are such that $R < 2\sqrt{\mathcal{L}/C}$, then u and v are both complex and

$u - v$ is imaginary and may be represented by $2j\beta$ (see Eqs. 12). Hence both q and i will be damped sine functions, as explained in Art. 3 above.

C. Limiting Case, $R = 2\sqrt{\mathcal{L}/C}$. — Eqs. (48) and (49) hold true only when $u \neq v$, that is, when the characteristics of the circuit are such as to produce either a dead-beat or an oscillatory transient. In the limiting case ($u = v$), K'_1 and K_2 in Eqs. (48) become infinite. Hence it is necessary to evaluate the constants A and B in Eqs. (44) which apply to the limiting case.

Assume that, after steady conditions have been established, a short-circuit occurs at the instant $t = 0$. In other words, when $t = 0$, $e = 0$, $q = (E_m/\omega Z) \cos(\alpha - \phi)$ and $i = E_m/Z \sin(\alpha - \phi)$. Substituting these values in Eq. (44) and solving for A and B we obtain:

$$\left. \begin{aligned} A &= (E_m/\omega Z) \cos(\alpha - \phi) & (a) \\ B &= (E_m/Z)[(R/2\mathcal{L}\omega) \cos(\alpha - \phi) - \sin(\alpha - \phi)] & (b) \end{aligned} \right\} \quad (50)$$

Hence the equations for charge and current in the limiting case are:

$$\left. \begin{aligned} q &= e^{-Rt/2\mathcal{L}}[A + Bt] & (a) \\ i &= (R/2\mathcal{L})[A + Bt]e^{-Rt/2\mathcal{L}} + Be^{-Rt/2\mathcal{L}} & (b) \end{aligned} \right\} \quad (51)$$

where A and B are expressed by Eqs. (50).

PROBLEMS

1. Cite instances in laboratory practice where circuits with R , $\mathcal{L}$, and C do occur.
2. Study Theorem VI Appendix I and make sure you understand each step in the Proof.

3a. A series circuit consists of a resistance $R = 100$ ohms, inductance $\mathcal{L} = 0.15$ mh., and capacitance $C = 15 \mu f$. This circuit is suddenly subjected to a continuous Emf. $E = 50$ volts.

- (a) Draw the time-current curve.
- (b) Draw the time-charge curve.

3b. A series circuit consisting of a resistance $R = 10$ ohms, inductance $\mathcal{L} = 1.5$ mh., and capacitance $C = 2 \mu f$ is suddenly subjected to a continuous Emf. $E = 100$ volts.

- (1) Compute the angular velocity β of the oscillatory charge and oscillatory current.
- (2) If $\beta = 2\pi f$ what will be the frequency of the natural oscillations?
- (3) Determine the expression for f in terms of the characteristics of the circuit and compare the result so obtained with Eq. (39), Chapter VII.
- (4) Compute the amplitudes of the damped sine waves of current and charge for three successive half cycles, and plot the curves of current and charge.
- (5) Determine the ratio of two successive amplitudes of the current curve, and compare this ratio with the term $e^{-Rt/2\mathcal{L}}$ where T is the period of the free oscillations.

3c. A series circuit having a resistance $R = 20$ ohms, inductance $\mathcal{L} = 0.3$ mh., and capacitance $C = 3 \mu f$ is suddenly subjected to a continuous Emf. of 110 volts.

- (a) Compute at least five points on the current and charge curves.
- (b) Plot the charge and current curves and compare them with the corresponding curves obtained in problem 3a.

3d. A series circuit consists of a capacitance an inductance (no-resistance), and a continuous Emf. E .

- (a) Show that this circuit falls under circuits discussed in case B, Art. 3.
- (b) Show that the charge and current curves are not damped but that the charge curve consists of a constant term and a sine function while the current curve is simply a sine function.
- (c) If such a resistanceless circuit could be obtained would it be possible to make a device which changes continuous to alternating current? Explain.
- (d) Determine the natural frequency of oscillations in such a circuit and compare your results with the expression for resonance given in Eq. (39), Chapter VII.

4a. After steady conditions have been established in the circuit of problem 3a, the source is suddenly short-circuited. Compute at least five values of the charge and current and plot the charge and current curves.

4b. The Emf. source in problem 3b is suddenly short-circuited after steady conditions have been established. Plot the discharge and current curve.

4c. After the condenser in problem 3c has been fully charged, the Emf. source is suddenly short-circuited. Give the curves of the discharge and current.

5a. A sine voltage of amplitude $E_m = 150$ volts and frequency $f = 60$ cycles per second is impressed on the series circuit specified in problem 3a. Moreover, at the time the circuit is established the instantaneous value of the applied voltage is a positive maximum. Draw to scale the voltage, current, and charge curves, considering the transient state.

5b. A sine voltage having an amplitude $E_m = 20$ volts and a frequency $f = 7000$ cycles per second was applied to the circuit specified in problem 3b. Moreover, the circuit was established at the instant when the voltage had a negative maximum.

- (1) Draw the current curve, considering the transient state.
- (2) Check the frequency and decrement of the oscillatory transient, obtained in this case, with the frequency and decrement of the transient obtained in problem 3c.

5c. The sine voltage specified in problem 5b was applied to the circuit given in problem 3c. Furthermore the voltage had an instantaneous value $e_0 = 5$ volts and a negative slope, at the instant of establishing the circuit. Compute at least three points on the transient current curve.

6. Explain physically why the transient state in a series circuit with R , $\mathcal{L}$, C and a given Emf. e is unavoidable.

7a. Assume the circuit specified in problem 5a to have been short-circuited (after steady conditions were established) at such an instant that the current was at its negative maximum. Plot the curve of the transient current that would be obtained under these conditions.

7b. Assume that, after steady conditions have been established in the circuit specified in problem 5b, a short-circuit occurs at such an instant that the current is at its zero value. What will be the transient current obtained?

7c. Let a short-circuit occur (after steady conditions have been established) in the circuit given in problem 5c, at such an instant that the voltage is at its positive maximum. Compute three points on the transient current curve.

CHAPTER XIX

CIRCUITS WITH UNIFORMLY DISTRIBUTED CHARACTERISTICS

1. Introduction. — Heretofore we have considered circuits with concentrated (lumped) characteristics. The resistance, inductance and capacitance of a circuit were regarded as being localized at just one part of the circuit. Moreover, a certain part of a circuit was occupied by one and only one characteristic. Strictly speaking this assumption is not valid even in the simplest circuit which consists of a straight conducting wire. Indeed the resistance of such a wire is distributed throughout all its length. The magnetic field, which encircles the current and by virtue of which the wire possesses the property of inductance, consists of tubes of flux which surround the entire length of the wire. Hence what we consider as concentrated inductance is really an inductance distributed over the whole length of the wire. Finally the capacitance of such a simple conductor comprises the capacitance of each element of length and is a distributed property.

For most purposes the distributed characteristics may be replaced by concentrated characteristics. In fact the laws established in the foregoing chapters apply to a large group of electric circuits commonly met with in practice. In some cases, however, it becomes essential to take account of the fact that the properties are distributed. Such is the case with a long distance transmission line, which is treated in this chapter not with a view of entering into the subject of electric power transmission but rather because it furnishes a practical example of this type of circuits.

2. The Distributed Properties of a Transmission Line. — Electric power is generally transmitted three-phase. If the polyphase circuit is balanced, then (see Art. 9, Chapter XV) the computations may be performed for a single line. Hence in what follows it will be assumed that the three-phase circuit is a star-star circuit; and that the phase Emf. is the Emf. from any line to the neutral and that the line-current is identical with the phase-current.

Each of the three phases of a transmission line possesses a distributed resistance, a distributed inductance, a distributed capacitance, and a distributed leakage conductance (or leakage).

By the term uniformly distributed characteristics is meant:

- (1) That the characteristics of a transmission line are to be considered as existing along the entire length of the line so that each element of length Δs (see Fig. 1) contains a certain amount of resistance ΔR , inductance $\Delta \mathcal{L}$, capacitance ΔC , and leakage conductance ΔG .
- (2) The characteristics of any one element Δs of the line are equal to those of any other element.

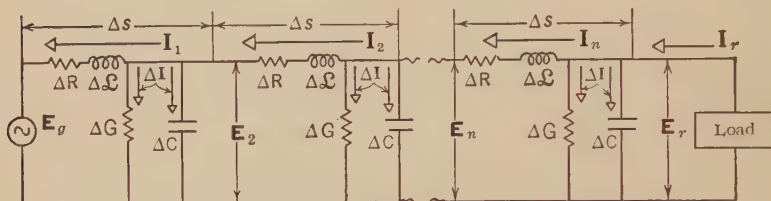


FIG. 1. The distributed properties of a transmission line.

We shall now define these characteristics of a transmission line.

(a) *Distributed Resistance.*—By the term resistance is meant the ohmic resistance of a conductor. This resistance, in the case of transmission lines, is uniformly distributed over the entire length of the conductor so that if the resistance of the conductor *per unit length* is R , then the resistance of an element of length ds is:

$$dR = R ds \quad (1)$$

(b) *Distributed Inductance.*—The inductance of any one conductor in a transmission line is due not only to the linkage of the current with the flux it produces but also to the linkage of that current with the flux produced by the neighboring conductors. In other words, the term inductance is here used to include the self-inductance per phase as well as the mutual inductance of the other phases. A thorough consideration of the inductance of transmission lines properly belongs to a treatise dealing with magnetic phenomena. Suffice it to say that tables have been computed giving the inductive reactance per conductor per unit length. These tables are given in Appendix II. Let the inductance, *per conductor, per unit length* of the line be $\mathcal{L}$ then the inductance of an element of length ds is:

$$d\mathcal{L} = \mathcal{L} ds \quad (2)$$

(c) *Distributed Capacitance.*—The capacitance of any one conductor in a transmission line consists of its ability to store up charge (1) due to its proximity to other conductors, and (2) due to its proximity to

the earth. Let the net capacitance *per conductor per unit length* of the line be C ; then the capacitance of an element of length ds is:

$$dC = C ds \quad (3)$$

(d) *Leakage Conductance.* — Owing to the high voltages used in power transmission, current may leak from one conductor to the air, or to another, or to the earth. The first two types of leakage occur due to corona formation. The last is due to creepage over the insulators, or to the actual conduction of current through faulty insulators, or to both factors. When corona formation occurs over the whole line the leakage is uniformly distributed. The leakage over insulators is, however, localized only where insulators exist (as on towers). Since distance between towers is generally small compared with the whole length of the line, leakage conductance may be considered as a uniformly distributed property of a transmission line. The conducting paths are represented by the conductances ΔG , Fig. 1. Let the conductance *per unit length* be G . Then the conductance of an element of length ds is:

$$dG = G ds \quad (4)$$

3. Current and Voltage Relations in a Circuit with Distributed Characteristics. — (a) *Physical Considerations.* — In order to appreciate what takes place physically in a circuit with distributed characteristics let the circuit, Fig. 1, be subdivided into N elements each having a length Δs and let each of these elements be considered as a circuit with concentrated characteristics, that is, a circuit having a resistance ΔR , inductance $\Delta \mathcal{L}$, conductance ΔG , and capacitance ΔC . Referring to Fig. 1 it will be observed that the current I_1 flowing through the first element is not equal to the current I_2 flowing through the second element because part of the current I_1 leaks through the conductance ΔG while another part charges the capacitance ΔC . The same may be stated in regard to the next two adjacent elements. Hence the current flowing through the line is not uniform throughout but varies as we move from one element to the next. In other words the current I is a function of the distance s along the line.

The same holds true of the Emf. between the line and the neutral. Thus the Emf. E_g , across the first element, Fig. 1, is not equal to the Emf. E_2 across the second element because part of the Emf. E_g is lost as a drop in potential across the resistance ΔR and inductance $\Delta \mathcal{L}$ of the first element. Similarly the Emf. E_2 across the second element is not the same as the Emf. E_3 across the third element. Hence the Emf. also varies as we move from one element to the next and may be considered as a function of the distance s along the line.

(b) *Mathematical Theory.*— Let the rate of change of Emf. with distance along the line be $\partial e/\partial s$;^{*} then the drop across an element of length ds is $(\partial e/\partial s)ds$. Let the average value of the current flowing through the element ds be i .[†] Then by Kirchhoff's Emf. law we have:

$$(\partial e/\partial s)ds = (dR)i + (d\mathcal{L})(\partial i/\partial t)^* \quad (5)$$

Substituting the values of dR and $d\mathcal{L}$ from Eqs. (1) and (2) respectively in Eq. (5) we have:

$$(\partial e/\partial s)ds = iR ds + \mathcal{L} ds (\partial i/\partial t) \quad (6)$$

Canceling ds we obtain:

$$(\partial e/\partial s) = Ri + \mathcal{L} (\partial i/\partial t) \quad (7)$$

Let the average voltage between the element ds and the neutral be e [‡] and let the rate of change of current along the line be $(\partial i/\partial s)$. The total change of current in the element ds is $(\partial i/\partial s)ds$. This change in current is due partly to leakage through the conductance and partly to current supplied to the capacitance. Hence by Kirchhoff's current law:

$$(\partial i/\partial s)ds = (dG)e + (dC)(\partial e/\partial t) \quad (8)$$

Substituting Eqs. (3) and (4) in Eq. (8) we obtain:

$$(\partial i/\partial s)ds = (Ge)ds + C(\partial e/\partial t)ds \quad (9)$$

whence:

$$(\partial i/\partial s) = Ge + C (\partial e/\partial t) \quad (10)$$

Eq. (7) states that the change in voltage per unit length is equal to the sum of the resistance and inductance drops per unit length.

Eq. (10) states that the change in current per unit length is equal to the sum (per unit length) of the current which leaks through the conductance and the current which is supplied to charge the line.

* Round ∂ signifies partial derivative. It is used because e and i are not only functions of distance but may also be functions of time, as indeed they are in the case of A.C. transmission.

† Strictly speaking, the value of the current changes along the element ds because by assumption some of the current leaks through the conductance while another part of it goes to charge the capacitance C . As ds approaches zero the actual value of the current is practically the same throughout the element and may be taken as the average value *in space* of the current flowing through the element.

‡ The value of the voltage changes along the element ds because of the resistance and inductance drops. In the limit, however, the values of these drops are so small that the voltage at the beginning of the element ds is practically identical with the Emf. at the end of it. Hence the average value *in space* of the Emf. along the element represents, to a fair degree, of accuracy, the Emf. across dG and dC .

Both Eqs. (7) and (10) are general because no assumptions have been made, thus far, as to their functional relations to time or to space.

These equations admit of an infinite number of solutions, in their most general form. However if e and i are assumed to be sine functions of time, and if the transient state is neglected, then $\partial e/\partial t$ and $\partial i/\partial t$ become known.

For the purpose of this treatment we shall assume that e and i are sine functions of time, so that:

$$\left. \begin{aligned} e &= E_m \sin (\omega t + \alpha) & (a) \\ i &= I_m \sin (\omega t + \beta) & (b) \end{aligned} \right\} \quad (11)$$

Substituting Eqs. (11) in (7) and (10) respectively we have:

$$\left. \begin{aligned} \partial [E_m \sin (\omega t + \alpha)] / \partial s &= R I_m \sin (\omega t + \beta) + \mathcal{L} \omega I_m \cos (\omega t + \beta) & (a) \\ \partial [I_m \sin (\omega t + \beta)] / \partial s &= G E_m \sin (\omega t + \alpha) + C \omega E_m \cos (\omega t + \alpha) & (b) \end{aligned} \right\} \quad (12)$$

Casting Eqs. (12) into the vector form we obtain:

$$\left. \begin{aligned} (\partial \mathbf{E} / \partial s) &= \mathbf{Z} \mathbf{I} & (a) \\ (\partial \mathbf{I} / \partial s) &= \mathbf{Y} \mathbf{E} & (b) \end{aligned} \right\} \quad (13)$$

where:

$$\left. \begin{aligned} \mathbf{Z} &= R + j\omega \mathcal{L} & * (a) \\ \mathbf{Y} &= G + j\omega C & * (b) \\ \mathbf{I} &= (I_m / \sqrt{2}) \epsilon^{j\beta} & (c) \\ \mathbf{E} &= (E_m / \sqrt{2}) \epsilon^{j\alpha} & (d) \end{aligned} \right\} \quad (14)$$

Eqs. (13) are two simultaneous differential equations in $\mathbf{E}$ and $\mathbf{I}$. One of the variables may be eliminated through differentiation. Thus differentiating Eq. (13a) with respect to s we have:

$$\partial^2 \mathbf{E} / \partial s^2 = \mathbf{Z} (\partial \mathbf{I} / \partial s) \quad (15)$$

Substituting in Eq. (15) the value of $(\partial \mathbf{I} / \partial s)$ as given in Eq. (13b) we obtain:

$$\partial^2 \mathbf{E} / \partial s^2 = \mathbf{Z} \mathbf{Y} \mathbf{E} = \mathbf{K}^2 \mathbf{E} \quad (16)$$

Similarly differentiating Eq. (13b) with respect to s and substituting in the resulting expression the value of $\partial \mathbf{E} / \partial s$ from Eq. (13a) we obtain:

$$\partial^2 \mathbf{I} / \partial s^2 = \mathbf{Z} \mathbf{Y} \mathbf{I} = \mathbf{K}^2 \mathbf{I} \quad (17)$$

where:

$$\mathbf{K}^2 = \mathbf{Z} \mathbf{Y} \quad (18)$$

Eqs. (16) and (17) are differential equations of the second order and resemble Eq. (6.1), Appendix I. Hence their general solution is given by Eq. (6.2), Appendix I.

* In Eq. (14) $\mathbf{Z} \neq 1/\mathbf{Y}$.

Comparing Eqs. (16) and (17) with Eq. (6.1), Appendix I, we find that $\mathbf{E}$ and $\mathbf{I}$ correspond to x , s corresponds to t , ab corresponds to $\bar{K}^2$, $a = 0$, and $h = 0$, hence (see Eq. (6.7), Appendix I) $u = \bar{K}$ and $v = -\bar{K}$.

Substituting these values in Eq. (6.2), Appendix I, and integrating we obtain:

$$\left. \begin{aligned} \mathbf{E} &= \mathbf{A}\epsilon^{-\bar{K}s} + \mathbf{B}\epsilon^{\bar{K}s} & (a) \\ \mathbf{I} &= \mathbf{D}\epsilon^{-\bar{K}s} + \mathbf{F}\epsilon^{\bar{K}s} & (b) \end{aligned} \right\} \quad (19)$$

In order to determine the four constants in Eq. (19) we shall agree to measure the distance s from the receiver (or load) end of the line,* so that at the load end $s = 0$. We shall further assume that the receiver vector voltage $\mathbf{E}_r$ and the receiver vector current $\mathbf{I}_r$ are known; namely, at $s = 0$, $\mathbf{E} = \mathbf{E}_r$ and $\mathbf{I} = \mathbf{I}_r$. Substituting these values in Eqs. (19) we obtain:

$$\left. \begin{aligned} \mathbf{E}_r &= \mathbf{A} + \mathbf{B} & (a) \\ \mathbf{I}_r &= \mathbf{D} + \mathbf{F} & (b) \end{aligned} \right\} \quad (20)$$

Here we have two equations with four constants. Hence solution of the problem is either impossible or there exist other physical conditions which have not been considered. These physical conditions are expressed by Eqs. (13). Thus differentiating Eqs. (19) with respect to s and substituting Eqs. (13) in the result we have:

$$\left. \begin{aligned} (\partial \mathbf{E} / \partial s) &= \bar{Z} \mathbf{I} = -\bar{K} \mathbf{A} \epsilon^{-\bar{K}s} + \bar{K} \mathbf{B} \epsilon^{\bar{K}s} & (a) \\ (\partial \mathbf{I} / \partial s) &= \bar{Y} \mathbf{E} = -\bar{K} \mathbf{D} \epsilon^{-\bar{K}s} + \bar{K} \mathbf{F} \epsilon^{\bar{K}s} & (b) \end{aligned} \right\} \quad (21)$$

Now at $s = 0$, $\mathbf{E} = \mathbf{E}_r$ and $\mathbf{I} = \mathbf{I}_r$. Hence Eqs. (21) become:

$$\left. \begin{aligned} \bar{Z} \mathbf{I}_r &= -\bar{K} \mathbf{A} + \bar{K} \mathbf{B} & (a) \\ \bar{Y} \mathbf{E}_r &= -\bar{K} \mathbf{D} + \bar{K} \mathbf{F} & (b) \end{aligned} \right\} \quad (22)$$

Solving Eqs. (20) and (22) simultaneously for the four constants $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{F}$ we obtain:

$$\left. \begin{aligned} \mathbf{A} &= 0.5 [\mathbf{E}_r - (\bar{Z}/\bar{K}) \mathbf{I}_r] & (a) \\ \mathbf{B} &= 0.5 [\mathbf{E}_r + (\bar{Z}/\bar{K}) \mathbf{I}_r] & (b) \\ \mathbf{D} &= 0.5 [\mathbf{I}_r - (\bar{Y}/\bar{K}) \mathbf{E}_r] & (c) \\ \mathbf{F} &= 0.5 [\mathbf{I}_r + (\bar{Y}/\bar{K}) \mathbf{E}_r] & (d) \end{aligned} \right\} \quad (23)$$

Substituting the value of $\bar{K}$ from Eq. (18) in Eq. (23) and then substituting Eqs. (23) in Eqs. (19) and rearranging terms we obtain:

$$\left. \begin{aligned} \mathbf{E} &= 0.5 \mathbf{E}_r (\epsilon^{\bar{K}s} + \epsilon^{-\bar{K}s}) + 0.5 \sqrt{(\bar{Z}/\bar{Y})} \mathbf{I}_r (\epsilon^{\bar{K}s} - \epsilon^{-\bar{K}s}) & (a) \\ \mathbf{I} &= 0.5 \mathbf{I}_r (\epsilon^{\bar{K}s} + \epsilon^{-\bar{K}s}) + 0.5 \sqrt{(\bar{Y}/\bar{Z})} \mathbf{E}_r (\epsilon^{\bar{K}s} - \epsilon^{-\bar{K}s}) & (b) \end{aligned} \right\} \quad (24)$$

* Here s is to be measured from the receiver end in such units of length as were taken in expressing the values of the line characteristics $\bar{Z}$ and $\bar{Y}$. Thus if $\bar{Z}$ and $\bar{Y}$ are expressed in ohms and mhos per mile, then s is in miles.

The functions $0.5 (\epsilon^{\dot{K}s} + \epsilon^{-\dot{K}s})$ and $0.5 (\epsilon^{\dot{K}s} - \epsilon^{-\dot{K}s})$ are of such practical importance that they have been given names. The first is called the *hyperbolic cosine* of $\dot{K}s$ (abbreviated $\cosh \dot{K}s$) and the second is known as the *hyperbolic sine* of $\dot{K}s$ (abbreviated $\sinh \dot{K}s$). Furthermore tables have been prepared giving the value of the functions corresponding to the argument $\dot{K}s$ when $\dot{K}$ is real. These tables are reproduced in Appendix II. Making these substitutions in Eq. (24) we obtain:

$$\left. \begin{aligned} \mathbf{E} &= \mathbf{E}_r \cosh \dot{K}s + Z_0 \mathbf{I}_r \sinh \dot{K}s & (a) \\ \mathbf{I} &= \mathbf{I}_r \cosh \dot{K}s + Y_0 \mathbf{E}_r \sinh \dot{K}s & (b) \end{aligned} \right\} \quad (25)$$

where:

$$\left. \begin{aligned} \cosh \dot{K}s &= (\epsilon^{\dot{K}s} + \epsilon^{-\dot{K}s})/2 & (a) \\ \sinh \dot{K}s &= (\epsilon^{\dot{K}s} - \epsilon^{-\dot{K}s})/2 & (b) \\ Z_0 &= \sqrt{\dot{Z}/\dot{Y}} & (c) \\ Y_0 &= \sqrt{\dot{Y}/\dot{Z}} = 1/Z_0 & (d) \\ \dot{K} &= \sqrt{\dot{Y}\dot{Z}} = \delta + j\psi & (e) \end{aligned} \right\} \quad (26)$$

In Eqs. (26), Z_0 and Y_0 are called the surge impedance and surge admittance of the line and $\dot{K}$ is known as the propagation constant of the line.

Eqs. (24) and (25) are identical and give the vector values of the Emf. $\mathbf{E}$ and the current $\mathbf{I}$, at any distance s from the receiver end, in terms of the receiver voltage, receiver current, and the characteristics $\dot{Z}$ and $\dot{Y}$ of the line. It is sometimes convenient to express $\mathbf{E}$ and $\mathbf{I}$ in terms of the sender (or generating station) voltage $\mathbf{E}_g$ and current $\mathbf{I}_g$. In this case the distance s' is measured from the sender end of the line. The following expressions are obtained in a manner similar to that used in obtaining Eqs. (25):

$$\left. \begin{aligned} \mathbf{E} &= \mathbf{E}_g \cosh \dot{K}s' + Z_0 \mathbf{I}_g \sinh \dot{K}s' & (a) \\ \mathbf{I} &= \mathbf{I}_g \cosh \dot{K}s' + Y_0 \mathbf{E}_g \sinh \dot{K}s' & (b) \end{aligned} \right\} \quad (27)$$

where s' is the distance measured from the sender end of the line in such units of length as were taken in expressing the values of the line characteristics $\dot{Z}$ and $\dot{Y}$; that is, if $\dot{Z}$ and $\dot{Y}$ are expressed in ohms and mhos per mile, then s' is in miles.

4. The Generator Current and Voltage Loci in Circuits with Distributed Characteristics. — In predetermining the operation of a transmission line, under various loads and various power factors, the designer is sometimes faced with a problem of the following nature.

For a given receiver voltage $\mathbf{E}_r$ (taken along the R -axis) it is required to determine the (sender) or generator voltage $\mathbf{E}_g$ and the current $\mathbf{I}_g$ as the receiver K.V.A. and receiver P.f. are varied. Theoretically this

involves a great deal of computations using Eqs. (25). The number of computations can, however, be reduced to a minimum through an analysis of Eqs. (25). Thus consider Eq. (25a). The distance s , under these conditions, is the length L of the line, which is a constant. Since $\mathbf{E}_r$ is a given constant therefore $\mathbf{E}_r \cosh KL$ is a constant and may be replaced by $(E_1 + jE_2)$. Again, since $Z_0 \sinh KL$ is a constant it can be replaced by $Z' \epsilon^{j\theta}$. Hence Eq. (25) becomes:

$$\left. \begin{aligned} \mathbf{E}_g &= E_1 + jE_2 + I_r \epsilon^{j\phi} Z' \epsilon^{j\theta} & (a) \\ &= E_1 + jE_2 + I_r Z' \epsilon^{j(\theta+\phi)} & (b) \end{aligned} \right\} \quad (28)$$

where ϕ is the phase angle between the receiver voltage and receiver current.

Now consider the case where the K.V.A. at the receiver end is zero. This means that $I_r = 0$ and hence $\mathbf{E}_g = E_1 + jE_2$. This is represented by the point P , Fig. 2, where E_1 is plotted along the R -axis and E_2 along the j -axis.

Next consider full load K.V.A. at the receiver end. Assume 100 per cent P.f. ($\phi = 0$) and compute $\mathbf{E}_g$ by using Eq. (25a). This gives the point a , Fig. 2. Since I_r at this load is constant and since ϕ may assume values varying from $+90^\circ$ to -90° depending upon the P.f. at the receiver end, the locus of the term $I_r Z' \epsilon^{j(\theta+\phi)}$ will be the solid semi-circle (marked 100%, Fig. 2) having a radius Pa equal to $I_r Z'$.

Now as the load increases or decreases in increments of 10 per cent the radii of the circles (marked 10, 20 . . . 110%) increase or decrease in equal increments. Hence no further computations are necessary. Simply draw these circles equally spaced from each other and having a common center point P . These circles are the loci of the terminus of $\mathbf{E}_g$ when the K. V.A. at the receiver end varies in steps of 10 per cent. Now it is convenient to know what the value of $\mathbf{E}_g$ would be at a certain specified P.f. The line Pa , Fig. 2, is drawn for 100 per cent P.f., evidently, as the P.f. increases in the leading or lagging direction ϕ increases. Hence look up the value of ϕ corresponding to a certain P.f. and draw radial lines marked 0.9, 0.85, etc., to the right and left of the radial line Pa (Fig. 2) making the angle $\pm \phi$ with the line Pa . The intersection of any radial line and circle represents the terminus of the sender voltage $\mathbf{E}_g$ at the specified K.V.A. and P.f. For example, if the sender voltage is required at 100 per cent K.V.A. (see circle marked 100%) and 90 per cent P.f. lagging (radial line 0.9 lag) then the sender voltage will be the vector $\mathbf{E}_g$ (Fig. 2) having an origin at the point 0 and a terminus at the intersection of circle 100 per cent and radial line 0.9 lag.

NOTE:—Radial lines are loci for constant

P. f. variable K. V. A.

Circles are loci for constant

K. V. A. variable P. f.

Full radial lines and circles

are voltage loci.

Dash-and-dot radial lines

and circles are current loci.

EXAMPLE:—For full load (100% K. V. A.)

90% P. f. Lagging.

$E_g = 136.5 + j54$ K. V.

$I_g = 192 + j104$ Amps.

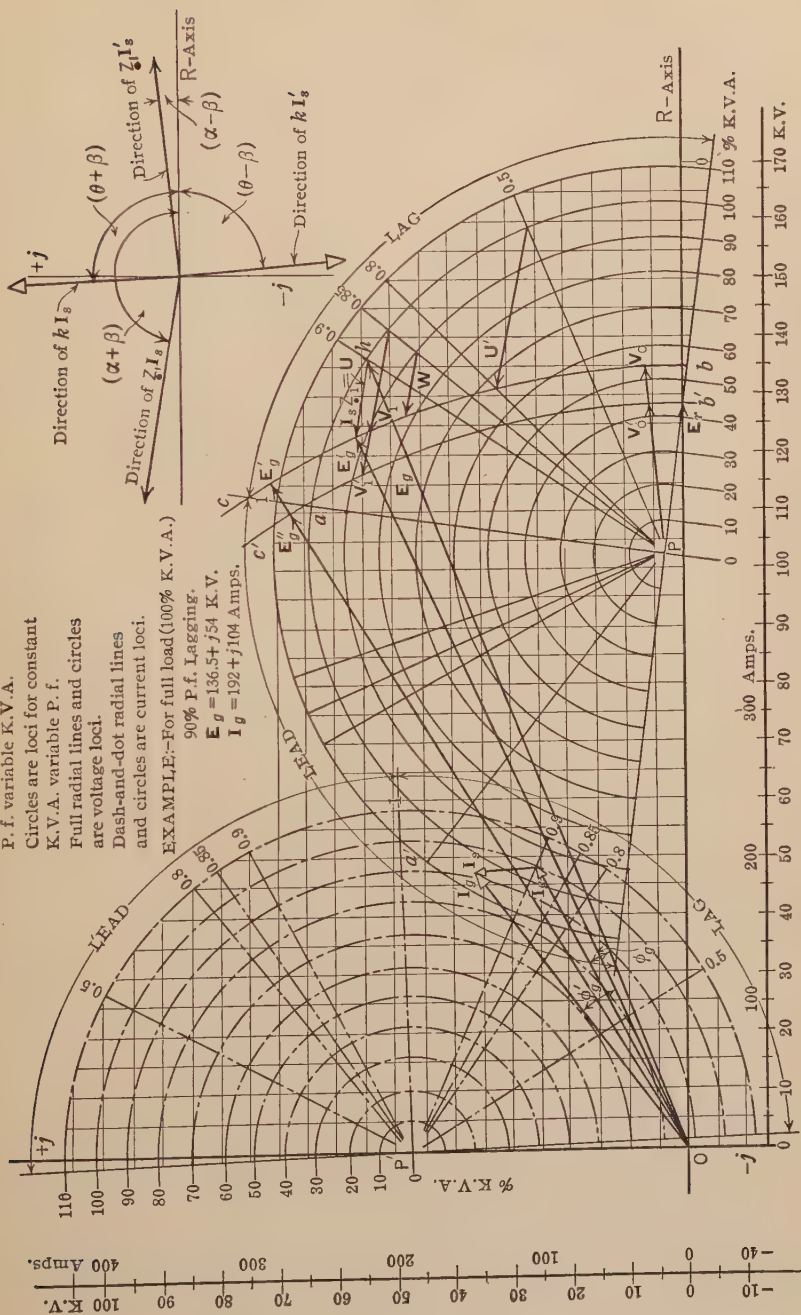


Fig. 2. Loci of the voltage and current vectors at the generator end of a transmission line.

It will be observed that only two computations are necessary to draw the locus diagram of the sender voltage. These are for zero K.V.A. which gives the point P and for 100 per cent K.V.A., unity P.f., which gives the point a . The remaining circles are merely drawn at distances equal to $(Pa/10)$ from each other and the radial lines are drawn making angles ϕ with line Pa .

Next Eq. (25b) is exactly of the same form as Eq. (25a); therefore the current loci are circles and straight lines as shown by the dotted part of Fig. 2. Here again only two computations are necessary to determine the points P' and a' . In order to determine P' take $I_r = 0$, $s = L$, and compute $Y_0 E_r \sinh mL$; in order to find a' take I_r equal to 100 per cent of the load current at 100 per cent P.f. and s equal to L , and compute I_g . Then the construction of the dotted circles and straight lines is identical with that described above.

Now suppose that it is required to determine from this diagram the value of the sender current I_g at 100 per cent K.V.A. and 90 per cent lagging P.f. Draw the vector I_g originating from point 0 and terminating at the intersection of circles 100 per cent K.V.A. and radial line 0.9 lag.

The advantages of this diagram are:

- (a) The ease with which it is constructed — It consists of nothing but circles and straight lines.
- (b) The minimum amount of labor required to make it — Only four computations.
- (c) The graphic representation of E_g , I_g and ϕ_g in the R - j plane.
- (d) The fair degree of accuracy it gives provided it is drawn to a large enough scale.

The only limitations imposed upon this graphical representation are:

- (1) It holds true for a given constant receiver voltage taken at the reference axis.
- (2) It holds true only for a given line with given characteristics and length.

An interesting fact is clearly demonstrated by Fig. 2, namely, that even at no load (point P') the current supplied by the generating station is an appreciable current (OP'). This current goes exclusively to charge the line. Its phase position relative to the voltage vector (OP) shows that the current is mainly reactive. As the load increases there exists a value of the load (see intersection of current circle 90 per cent K.V.A. and radial line 0 lag) at which the current supplied by the generator is almost negligible. Thus, although the load is drawing 90 per

cent K.V.A., at zero P.f. lagging, still the K.V.A. supplied by the generator is very small indeed. This does not imply that more power is drawn by the load than is supplied by the source. Indeed from the principle of conservation of energy the power supplied by the source must be identical with the power consumed by the line and load. A few of the many striking (indeed apparently anomalous) facts that are revealed by this diagram are here pointed out in order to encourage the reader to subject Fig. 2 to a more scrutinizing examination. In problems 4 at the end of the chapter, the reader's attention is directed to other salient facts regarding the behavior of a circuit with distributed characteristics.

5. Operation of Transmission Lines at Constant Voltage.— It is sometimes desirable to maintain the effective values of the receiver and sender voltages constant irrespective of any variation in the load. This is generally done by installing at the load end a synchronous motor (condenser) which draws a current I_s at about 4 per cent P.f. leading or lagging. Regulators are also installed at the generator end to control the exciting current. Under these conditions Eqs. (25) become:

$$\left. \begin{aligned} \mathbf{E}'_g &= \mathbf{E}_r \cosh \bar{K}L + Z_0(\mathbf{I}_r + \mathbf{I}_s) \sinh \bar{K}L & (a) \\ \mathbf{I}'_g &= (\mathbf{I}_r + \mathbf{I}_s) \cosh \bar{K}L + Y_0\mathbf{E}_r \sinh \bar{K}L & (b) \end{aligned} \right\} \quad (29)$$

where:

$\mathbf{E}'_g$ is the generator voltage. Its effective value is constant but its phase position is not necessarily constant. In other words the vector $\mathbf{E}'_g$ has a constant length (modulus) but a variable argument.

$\mathbf{E}_r$ is the receiver voltage. It is constant both in magnitude and phase position and is taken along the reference axis.

$\mathbf{I}'_g$ is the generator current. It is a vector which varies both in magnitude and in phase position.

$\mathbf{I}_r$ is the receiver (load) current. It is a variable vector.

$\mathbf{I}_s$ is the synchronous condenser current. It is a vector which varies in magnitude but may assume one of two phase positions, that is, it is displaced from the R -axis by a given angle $\pm \beta$ so that $\mathbf{I}_s = I_s e^{\pm j\beta}$.

L is the total length of the line.

It is required (1) to compute the K.V.A. of the synchronous condenser corresponding to any load K.V.A., and (2) to predetermine the current and voltage vectors at the generator end.

We shall give both an analytical as well as a graphic solution of this problem.

A. Analytical Solution, (1) Determination of the K.V.A. of the Synchronous Condenser. — Since the synchronous condenser is connected at the load (receiver) end of the line, it operates at the receiver voltage. Hence its K.V.A. is $E_r I_s$. Now E_r is known. Consequently the problem of computing the K.V.A. of the synchronous condenser is reduced to the determination of I_s . In order to solve Eq. (29) for I_s let:

$$\left. \begin{aligned} \mathbf{E}_r \cosh \bar{KL} + Z_0 \mathbf{I}_r \sinh \bar{KL} &= E + jE' & (a) \\ \mathbf{E}'_g &= E'_g e^{j\alpha} & (b) \\ \mathbf{I}_s Z_0 \sinh \bar{KL} &= I_s (\cos \beta + j \sin \beta)(a + jb) = I_s(c + jd) & (c) \end{aligned} \right\} \quad (30)$$

Substituting these values in Eq. (29a) we have:

$$\left. \begin{aligned} E'_g e^{j\alpha} &= (E + cI_s) + j(E' + dI_s) & (a) \\ E'_g e^{-j\alpha} &= (E + cI_s) - j(E' + dI_s) & (b) \end{aligned} \right\} \quad (31)$$

whence:

Multiplying Eqs. (31a) and (31b), simplifying and rearranging terms we obtain:

$$\left. \begin{aligned} I_s^2(c^2 + d^2) + I_s(2cE + 2dE') + (E^2 + E'^2 - E'^2_g) &= 0 & (a) \\ \text{or: } AI_s^2 + BI_s + F &= 0 & (b) \end{aligned} \right\} \quad (32)$$

$$\text{whence:} \quad I_s = (-B \pm \sqrt{B^2 - 4AF})/2A \quad (33)$$

Eq. (33) shows that there exist two values of the synchronous condenser current which give the desired regulation. Naturally the smaller of the two values (if positive) is the required value of I_s .

(2) *Determination of $\mathbf{E}'_g$ and $\mathbf{I}'_g$.* — Having thus determined I_s we can compute the values of $\mathbf{E}'_g$ and $\mathbf{I}'_g$ by the use of Eqs. (29).

B. Graphic Solution. — By assumption, the effective value of the generator voltage E_g is to be maintained constant. Hence the locus of the terminus of E_g is a circle having a radius equal to the effective value E_g and a center at the origin 0 of the R - j plane. A part of this locus (the arc cb) is shown in Fig. 2.

Having fixed the length of the vector $\mathbf{E}_g$ it becomes necessary to determine its phase position. Eq. (29) shows that the vector $\mathbf{E}'_g$ consists of three components:

- (a) The vector $\mathbf{E}_r \cosh \bar{KL}$. This is the generator voltage at no load. It is represented by the line OP , Fig. 2.
- (b) The vector $\mathbf{I}_r Z_0 \sinh \bar{KL}$. This is the line drop due to the load current. It is represented by any vector drawn from the point P to any intersection of one of the circles and one of the radial line (as the line ph , Fig. 2).
- (c) The vector $\mathbf{I}_s Z_0 \sinh \bar{KL}$. This vector represents the drop in the line due to the synchronous condenser current. It is this vector which must be determined.

Let the drop in the line due to $\mathbf{I}_s$ be written in this form:

$$\left. \begin{aligned} \mathbf{I}_s Z_0 \sinh \bar{K}L &= I_s \epsilon^{\pm j\beta} Z_1 \epsilon^{j\alpha} = I_s Z_1 \epsilon^{j(\alpha \pm \beta)} = \mathbf{I}_s Z_1 & (a) \\ \text{where: } Z_1 &= Z_0 \sinh \bar{K}L & (b) \end{aligned} \right\} \quad (34)$$

Since, by assumption, β (the angle of lead or lag of the synchronous condenser current) is known and is maintained constant, and since α (which represents the argument of the product $Z_0 \sinh \bar{K}L$) is also constant, then the argument $\alpha \pm \beta$ is constant. Hence the two possible directions of the vector $\mathbf{I}_s Z_1$ are known. These are (1) a line making an angle $\alpha + \beta$ with the R -axis, and (2) a line making an angle $(\alpha - \beta)$ with the R -axis. These two lines are shown in Fig. 2 as $\mathbf{I}_s Z_1$ and $\mathbf{I}'_s Z_1$. The first, $\mathbf{I}_s Z_1$, represents the direction of the drop when the synchronous condenser draws a leading current. The second, $\mathbf{I}'_s Z_1$, represents the direction of the drop when the synchronous condenser draws a lagging current.

The magnitude of the vector $\mathbf{I}_s Z_1$ can be determined by referring to Eq. (29) and Fig. 2. Graphically the first term in this equation, that is, $\mathbf{E}_r \cosh \bar{K}L$, is represented by the vector OP . The second term, $\mathbf{I}_r Z_0 \sinh \bar{K}L$, is represented by the vector Ph ; and the third term, $\mathbf{I}_s Z_1$, is represented by the vector $\mathbf{I}_s Z_1$ whose origin is at h and whose terminus must lie on the circle bc which is the locus of $\mathbf{E}'_g$. Thus both the magnitude and direction of $\mathbf{I}_s Z_1$ are determined. Moreover the vector $\mathbf{E}'_g$ (Fig. 2) originates at the point 0 and terminates at the intersection of $\mathbf{I}_s Z_1$ and the circular locus cb .

In order to determine the vector $\mathbf{I}'_g$ (Eq. 29b) we note that this current consists of three components:

- (a) The charging and leaky currents, $Y_0 \mathbf{E}_r \sinh \bar{K}L$. This is represented by the vector OP' ; Fig. 2.
- (b) The load current $\mathbf{I}_r \cosh \bar{K}L$. This is represented by any vector drawn from the point P' to the intersection of any of the current circles and radial lines.
- (c) The synchronous condenser current, $\mathbf{I}_s \cosh \bar{K}L$, which is to be determined. The two phase positions of this current at the generator end are given and may be plotted by writing:

$$\mathbf{I}_s \cosh \bar{K}L = I_s \epsilon^{\pm j\beta} k \epsilon^{j\theta} = I_s k \epsilon^{j(\theta \pm \beta)} = k \mathbf{I}_s \quad (35)$$

These phase positions of the synchronous condenser current are shown in Fig. 2 for leading and lagging currents. The vectors $k\mathbf{I}_s$ and $k\mathbf{I}'_s$ are for leading and lagging currents respectively. As to its length, this can be easily determined as follows:

In Eq. (29) the currents I_r and I_s are multiplied by the same impedance operator $Z_0 \sinh KL$. Hence the ratio of the lengths of the vectors $I_s Z_1$ and Ph (Fig. 2) are as the ratio of the currents; or:

$$(I_s/I_r) = (I_s Z_1/Ph) \quad \text{or} \quad I_s = (I_s Z_1/Ph) I_r \quad (36)$$

But I_r is represented by a line drawn from P' to the intersection of any circle and radial line (for example, the line $P'd$). Its length is thus known. Hence the length of I_s may be found from Eq. (36) in terms of the three known moduli, I_r , $I_s Z_1$ and Ph .

It must be here observed that, if $I_s Z_1$ is in the direction of lead, then I_s (Fig. 2) must also be drawn leading; and vice versa.

Having thus determined I_s we add it to I_g and obtain I'_g , which is the generator current at the specified K.V.A., P.f., and constant load and generator voltages.

6. Minimum K.V.A. Synchronous Condenser. — Possibly the principal disadvantage of constant voltage operation is the cost of synchronous condensers. Hence anything that can be done to decrease their size and, therefore, their cost is well worth considering. We have already stated that the K.V.A. rating of a synchronous condenser is proportional to its current rating I_s because E_r is constant. Moreover, the value of I_s (see Fig. 2) is proportional to the length of the vector $I_s Z_1$. (See Eq. 36.)

Now the length of the vector $I_s Z_1$ varies with:

- (a) *The K.V.A. of the load.* Thus the vector $\mathbf{U}$ (Fig. 2) for 100 per cent K.V.A. is not equal to the vector $\mathbf{U}$ for 90 per cent K.V.A.
- (b) *The P.f. of the load.* Examine the vectors $\mathbf{U}$ and $\mathbf{U}'$ both of which are for the same K.V.A. but different power factors.
- (c) *The generator voltage selected.* Thus the length of the vector $\mathbf{V}_1$ will be different depending upon whether it terminates on the line bc or $b'c'$ both of which are constant voltage loci.

The first two of these three factors are fixed by the conditions at the receiver end. Furthermore they are outside the control of the designer because they are determined by the magnitude and nature of the consumer's load. Hence it remains to select such a generator voltage as to obtain constant voltage operation with the minimum size of synchronous condenser.

In order to make the problem more specific let it be required to determine the minimum synchronous condenser which would give constant generator voltage at no load as well as at 85 per cent lagging P.f., full load. This is accomplished by selecting such a constant generator voltage E'_g that the length of the vector $\mathbf{V}'_1$ (which represents $I_s Z_1$ at

full load) is equal to the modulus of the vector V'_0 (which represents $I'_s Z_1$ at no load). The locus of E'_g is then the circle $b'c'$, Fig. 2. To show that such is the case, assume that E'_g (whose locus is bc , Fig. 2) is the generator voltage which requires the minimum synchronous condenser. At full load the magnitude of the synchronous condenser current (and therefore the K.V.A. rating of the synchronous condenser) is decreased (see Fig. 2) in the ratio of (V_1/V'_1) . But at no load, the current I_s is increased in the ratio of (V_0/V'_0) . As the synchronous condenser must be designed for its maximum load, therefore at this new voltage the K.V.A. rating is increased in the ratio of (V_0/V'_0) . Hence the synchronous condenser required at the voltage E'_g is by no means the smallest synchronous condenser that could be used.

7. Variation of the Vector Voltage and Current with the Distance s Along the Line. — We have, so far, discussed the relationship between the voltage and the current at the receiver and generator ends of a long transmission line. It would be interesting to study the general variation of E and I with the distance s along a line. In order to gain a clear insight into this functional relationship we shall use Eqs. (24) and substitute in them the values of the constants as given in Eqs. (26). Thus:

$$\left. \begin{aligned} E &= 0.5 E_r (\epsilon^{\delta s} \epsilon^{j\psi s} + \epsilon^{-\delta s} \epsilon^{-j\psi s}) + 0.5 Z_0 I_r (\epsilon^{\delta s} \epsilon^{j\psi s} - \epsilon^{-\delta s} \epsilon^{-j\psi s}) \quad (a) \\ I &= 0.5 I_r (\epsilon^{\delta s} \epsilon^{j\psi s} + \epsilon^{-\delta s} \epsilon^{-j\psi s}) + 0.5 Y_0 E_r (\epsilon^{\delta s} \epsilon^{j\psi s} - \epsilon^{-\delta s} \epsilon^{-j\psi s}) \quad (b) \end{aligned} \right\} \quad (37)$$

or:

$$\left. \begin{aligned} E &= 0.5 (E_r + Z_0 I_r) \epsilon^{\delta s} \epsilon^{j\psi s} + 0.5 (E_r - Z_0 I_r) \epsilon^{-\delta s} \epsilon^{-j\psi s} \quad (a) \\ I &= 0.5 (I_r + Y_0 E_r) \epsilon^{\delta s} \epsilon^{j\psi s} + 0.5 (I_r - Y_0 E_r) \epsilon^{-\delta s} \epsilon^{-j\psi s} \quad (b) \end{aligned} \right\} \quad (37')$$

Let:

$$\left. \begin{aligned} 0.5 (E_r + Z_0 I_r) &= E_1 \epsilon^{j\theta} \quad (a) & 0.5 (E_r - Z_0 I_r) &= E_2 \epsilon^{j\gamma} \quad (b) \\ 0.5 (I_r + Y_0 E_r) &= I_1 \epsilon^{j\sigma} \quad (c) & 0.5 (I_r - Y_0 E_r) &= I_2 \epsilon^{j\eta} \quad (d) \end{aligned} \right\} \quad (38)$$

Substituting these values in Eq. (37') and simplifying we obtain:

$$\left. \begin{aligned} E &= E_1 \epsilon^{\delta s} \epsilon^{j(\psi s + \theta)} + E_2 \epsilon^{-\delta s} \epsilon^{-j(\psi s - \gamma)} = E_1 + E_2 \quad (a) \\ I &= I_1 \epsilon^{\delta s} \epsilon^{j(\psi s + \sigma)} + I_2 \epsilon^{-\delta s} \epsilon^{-j(\psi s - \eta)} = I_1 + I_2 \quad (b) \end{aligned} \right\} \quad (39)$$

Eq. (39a) shows that the Emf. at any distance s from the receiver end of a transmission line is the sum of two vectors one of which (E_1) has a modulus ($E_1 \epsilon^{\delta s}$) which increases exponentially as s increases and the other (E_2) has a modulus $E_2 \epsilon^{-\delta s}$ which decreases (or is attenuated) as s increases. The factor which accounts for this decrease is known as the *attenuation constant*.

The first vector, E_1 , has an argument $(\psi s + \theta)$ which increases positively as s increases, causing the vector to rotate counter-clockwise.

The second vector, $\mathbf{E}_2$, has an argument $-(\psi_s - \gamma)$ which increases negatively as s increases causing the vector $\mathbf{E}_2$ to rotate clockwise. The loci of these two vectors are respectively the two logarithmic spirals $\mathbf{E}_1$ and $\mathbf{E}_2$ (Fig. 3a); and the locus of the voltage $\mathbf{E}$ is the elliptical logarithmic spiral $\mathbf{E}$ (Fig. 3a). This spiral is obtained by simply adding the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ corresponding to the same distances.

Similarly the current $\mathbf{I}$ at any distance (Eq. 39b) is the sum of the two vectors $\mathbf{I}_1$ and $\mathbf{I}_2$ whose loci are the logarithmic spirals $\mathbf{I}_1$ and $\mathbf{I}_2$ (Fig. 3b). The locus of $\mathbf{I}$ is the elliptical spiral.

Figs. 3 show, at a glance, not only the magnitude of the vector voltage $\mathbf{E}$ and current $\mathbf{I}$ at any distance s along the line, but also the phase positions of $\mathbf{E}$ and $\mathbf{I}$. They thus offer a complete representation of the physical conditions prevailing at any distance s from the receiver end of the line. This figure is plotted for full load (200,000 K.V.A.) 80 per cent P.f. supplied by a line of infinite length having the following characteristics which also apply to Fig. 2:

$$Z = 0.114 + j0.792 \text{ ohm per mile.}$$

$$Y = 0 + j5.49 \times 10^{-6} \text{ mho per mile.}$$

The voltage from line to neutral at the receiver end is 127,000 volts and is in phase with the R -axis. No condenser operates at the load end.

Each of the divisions on the loci (Figs. 3) represents a distance of 100 miles. Hence the numbers marked 1, 2 . . . 8 represent 100, 200 . . . 800 miles respectively.

Let it be required to find the vector Emf. and current at a distance of 300 miles from the receiver end of the line. Draw the vectors $\mathbf{E}$ and $\mathbf{I}$ from the origin to the third division from the zero point on the elliptical locus (see Figs. 3) and read the projections of these vectors on the R -axis and j -axis. This gives approximately:

$$E = 147 + j47 = 154 \angle 17^\circ 40' \text{ K.v.}$$

$$I = 175 + j65 = 187 \angle 20^\circ 30' \text{ amp.}$$

Thus although the current at the receiver end of the line lags the voltage by an angle whose cosine is 0.8, it actually leads the voltage, at a distance of 300 miles from the receiver end, by an angle equal to $2^\circ 50'$. This change in phase is one of the conditions which are encountered in circuits with distributed characteristics.

8. The Instantaneous Value of the Voltage and Current in an Infinitely Long Circuit with Distributed Characteristics. — The Concept of Waves in Space. — Let the current and voltage elements of several oscillograms be properly connected at equal intervals along a transmission line of infinite length and let all these oscillograms be operated at

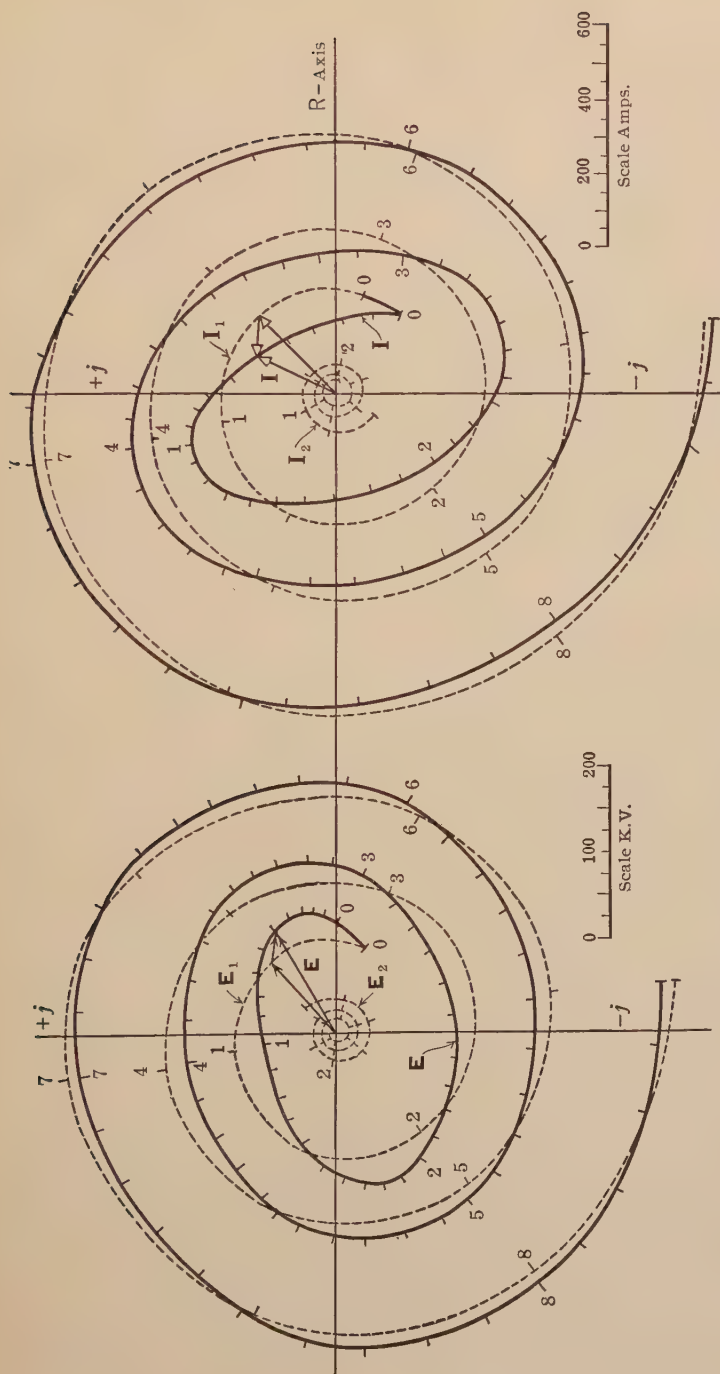


FIG. 3a. Locus of the Vector voltage as a function of the distance from the receiver end of a transmission line. FIG. 3b. Locus of the Vector current as a function of the distance from the receiver end of a transmission line.

one and the same instant $t = 0$ such that the instantaneous value of the Emf. at the receiver end is zero. It is required to predetermine the records given by these various oscillographs, that is, the instantaneous values of the Emf. and current at any distance s along the line for the instant $t = 0$. Such a problem can be solved by the use of Figs. 3 when it is recalled that the instantaneous value of a sine wave is equal to the j -component of the vector representing this sine wave (see Art. 12, Chapter III). Hence in order to determine the instantaneous values e and i at any point along the line read the value of the j -component of the vectors $\mathbf{E}$ and $\mathbf{I}$, Figs. 3, corresponding to that point. If this is done and the instantaneous values of the Emf. and current are plotted as ordinates corresponding to the distance s as abscissas, then the two waves e and i , Fig. 4, would be obtained.

The current and Emf. waves, Fig. 4, are incremental sine waves which have a definite periodicity and a definite wavelength λ . For the particular case under consideration, λ is approximately 3150 miles. It must be here emphasized that the waves, Fig. 4, are *not time* but *space waves*.

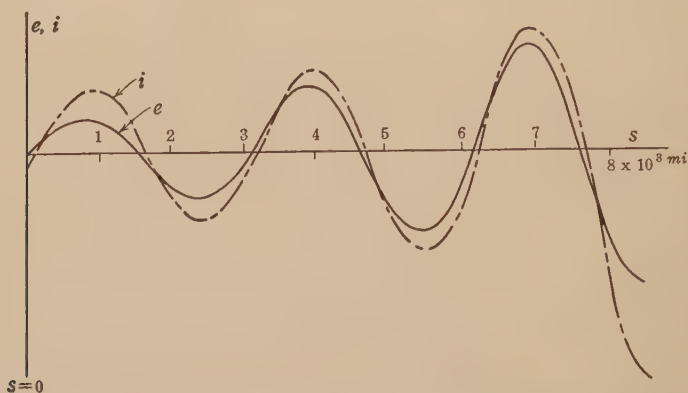


FIG. 4. Instantaneous values e and i as functions of the distance s measured from the receiver end of a transmission line.

9. Forced Oscillations of a Circuit with Distributed Characteristics. —

Consider a resilient rod, Fig. 5a. Let one end be free and the other end held by the hand. Let the hand holding the rod be moved briskly sideways. The rod will be found to oscillate (also sideways) and assume the shapes a and a' alternately. The rod is said to oscillate in a quarter wave. If the hand is shaken more briskly a point will be reached when the rod in oscillating takes alternately the shapes b and b' . The rod is now said to oscillate in a three-quarter wave. A more brisk oscilla-

tion of the hand causes the rod to oscillate in $(5/4)$, $(7/4)$, $(9/4)$. . . $(2n + 1)/4$ wave. In each case an amplitude of the oscillations always occurs at the free terminal of the rod. This mode of oscillation is called forced oscillation in contradistinction with free oscillation which occurs when the rod is given just one impulse and after that is left free to oscillate at its natural period.

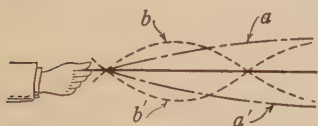


FIG. 5a.

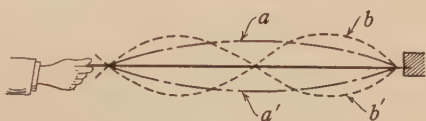


FIG. 5b.

FIG. 5a. Forced oscillations of a resilient rod, with free end, corresponding to the forced oscillations of a transmission line, Fig. 5c, carrying no load.

FIG. 5b. Forced oscillations of a resilient rod, with fastened end, corresponding to the forced oscillations of a loaded transmission line Fig. 5d.

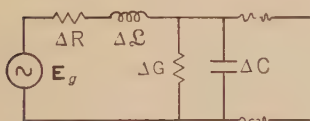


FIG. 5c. An unloaded transmission line.

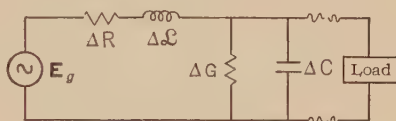


FIG. 5d. A loaded transmission line.

Next let a resilient rod (Fig. 5b) be fastened at one end while the other end is held by the hand; and let the operations described above be repeated. The rod would in this case oscillate in a half wave (curve *a*) or $(2/2)$ wave (curve *b*) or $(3/2)$, $(4/2)$, . . . $(n/2)$ wave, depending upon the briskness of the hand oscillations. In each case a node (zero point) occurs at the fastened end because this end is held fast and can not oscillate.

The resilient rod with an open end (Fig. 5a) corresponds to the transmission line without load (Fig. 5c). The oscillations of the hand correspond to the generator Emf. oscillations which are transmitted in space along the line. Similarly the fastened rod, Fig. 5b, corresponds to the loaded transmission line, Fig. 5d.

Though the analogy is not complete in all its details it brings out the following facts relative to the forced oscillations of a transmission line:

- (a) The forced oscillations of an open transmission line are different from those of a closed line. Consequently the study of forced oscillations divides itself into two main cases:
 - (1) Oscillations of an unloaded line.
 - (2) Oscillations of a loaded line.

- (b) In the unloaded line forced oscillations in space result in an anti-node (amplitude) at the receiver end whereas in a loaded line forced oscillations result in a node (zero value) at the receiver end.
- (c) As a consequence of (b), an unloaded transmission line oscillates in odd multiples of a quarter wavelength whereas a loaded line oscillates in integer multiples (odd or even) of a half wavelength.

We shall now develop numerical relations giving:

- (a) The length of line which would undergo forced oscillations at a given generator frequency.
- (b) The generator frequency at which a line of a given length would oscillate.

This frequency is generally known as the *natural or resonant frequency* f_r of a circuit with distributed characteristics.

10. The Wavelength λ of Space Waves and the Resonant Frequency of a Circuit with Distributed Characteristics. — Consider a circuit with distributed characteristics R , $\mathcal{L}$, G , and C , and let it be required to determine:

- (a) *The Wavelength λ of the Space Waves* when the circuit is subjected to a generator voltage of a frequency f under the following conditions:
- (1) The circuit is not loaded.
 - (2) The circuit is loaded.
- (b) *The Resonant Frequency, f_r , of the Circuit*, that is, such a frequency that, when impressed on a circuit of a given length L , it causes that circuit to undergo forced oscillations. Here again two conditions must be considered:
- (1) The circuit is not loaded.
 - (2) The circuit is loaded.

(a₁) *The Wavelength of Space Waves in an Unloaded Circuit.* — The condition that a transmission circuit carries no load is expressed by the fact that $I_r = 0$. Hence Eq. (37'a) becomes:

$$\mathbf{E} = 0.5 \mathbf{E}_r [\epsilon^{\delta s} \epsilon^{j\psi s} + \epsilon^{-\delta s} \epsilon^{-j\psi s}] \quad (40)$$

which reduces to the form:

$$E' \epsilon^{jk} = (\mathbf{E}/0.5 \mathbf{E}_r) = \epsilon^{\delta s} (\cos \psi s + j \sin \psi s) + \epsilon^{-\delta s} (\cos \psi s - j \sin \psi s) \quad (41)$$

whence the argument ζ of the complex number ($E/0.5 E_r$) is:

$$\tan \zeta = \sin \psi s (\epsilon^{\delta s} - \epsilon^{-\delta s}) / \cos \psi s (\epsilon^{\delta s} + \epsilon^{-\delta s}) \quad (42)$$

where δ and ψ are defined by Eq. (26e).

Now an unloaded circuit (Fig. 5a) undergoes forced oscillations when its length s corresponds to $(1/4), (3/4) \dots (q/4)$ of a wavelength. Mathematically this means that the angle ζ can be $(\pi/2), (3\pi/2) \dots (q\pi/2)$ where q is any odd integer.

But if $\zeta = q\pi/2$, then $\tan \zeta = \infty$. This can be the case either when the numerator of Eq. (42) is infinite or the denominator is zero. But the numerator can not be infinite so long as s is finite. Hence the denominator of Eq. (42) is zero; or:

$$\cos \psi s (\epsilon^{\delta s} + \epsilon^{-\delta s}) = 0 \quad (43)$$

Therefore

$$\left. \begin{aligned} \psi s &= (q\pi/2) & (a) \\ s &= (q\pi/2 \psi) & (b) \end{aligned} \right\} \quad (44)$$

where q is any odd integer and represents the number of quarter wavelengths λ which s comprises. Thus if $q = 1$, then s is $(1/4)\lambda$; if $q = 3$ then s is $(3/4)\lambda$, etc.

(a₂) *The Wavelength λ of Space Waves in a Loaded Circuit.* — Let Eq. (39a) be written in the following form:

$$\left. \begin{aligned} E\epsilon^{\zeta s} &= E_1\epsilon^{\delta s} [\cos(\psi s + \theta) + j \sin(\psi s + \theta)] \\ &\quad + E_2\epsilon^{-\delta s} [\cos(\psi s - \gamma) - j \sin(\psi s - \gamma)] & (a) \\ &= [E_1\epsilon^{\delta s} \cos(\psi s + \theta) + E_2\epsilon^{-\delta s} (\cos \psi s - \gamma)] \\ &\quad + j [E_1\epsilon^{\delta s} \sin(\psi s + \theta) - E_2\epsilon^{-\delta s} \sin(\psi s - \gamma)] & (b) \end{aligned} \right\} \quad (45)$$

Then

$$\tan \zeta = \frac{E_1\epsilon^{\delta s} \sin(\psi s + \theta) - E_2\epsilon^{-\delta s} \sin(\psi s - \gamma)}{E_1\epsilon^{\delta s} \cos(\psi s + \theta) + E_2\epsilon^{-\delta s} \cos(\psi s - \gamma)} \quad (46)$$

Now a loaded circuit (Fig. 5b) oscillates when its length s corresponds to $(1/2), (2/2) \dots (n/2)$ of the length of a space wave. Mathematically this means that the angle ζ in Eq. (46) is $\pi, 2\pi, 3\pi \dots n\pi$ where n is any integer. Consequently $\tan \zeta$ is zero. Such a condition is satisfied either when the denominator in Eq. (46) is infinite or when the numerator is zero. But the denominator can not be infinite as long as the length s of the line is finite. Hence, for forced oscillations, the numerator of Eq. (46) must be equal to zero; or:

$$E_1\epsilon^{2\delta s} \sin(\psi s + \theta) - E_2\epsilon^{-\delta s} \sin(\psi s - \gamma) = 0 \quad (47)$$

Dividing Eq. (47) throughout by $E_2\epsilon^{-\delta s}$ and rearranging terms we have:

$$(E_1/E_2)\epsilon^{2\delta s} = \sin(\psi s - \gamma) / \sin(\psi s + \theta) \quad (48)$$

$$\text{Let} \quad (\psi s + \theta) = x \quad \text{then} \quad s = (x - \theta)/\psi \quad (49)$$

Then Eq. (48) becomes:

$$\left. \begin{aligned} (E_1/E_2) e^{2\delta s} &= \sin(x - \theta - \gamma)/\sin x & (a) \\ &= [\sin x \cos(\theta + \gamma) - \cos x \sin(\theta + \gamma)]/\sin x & (b) \\ &= \cos(\theta + \gamma) - \cot x \sin(\theta + \gamma) & (c) \end{aligned} \right\} \quad (50)$$

Substituting the value of x from Eq. (49) in Eq. (50c) we have:

$$(E_1/E_2) e^{2\delta s} = \cos(\theta + \gamma) - \cot(\psi s + \theta) \sin(\theta + \gamma) \quad (51)$$

Eq. (51) can be solved graphically by plotting s as abscissa against $(E_1/E_2) e^{2\delta s}$ and $\cos(\theta + \gamma) - \cot(\psi s + \theta) \sin(\theta + \gamma)$ as ordinates. These curves (see Fig. 6) intersect at two points in the interval $0 < (\psi s + \theta) \leq 2\pi$. The abscissa of one point of intersection has a value corresponding to half a wavelength and thus satisfies Eq. (47). The other point of intersection has an abscissa s equal to a full wavelength. Other points of intersection occur beyond that interval at $s = (3\lambda/2)$, $(4\lambda/2) \dots, (n\lambda/2)$. The points of intersection give the required values of s which result in oscillations and consequently specify the wavelength λ of the line.

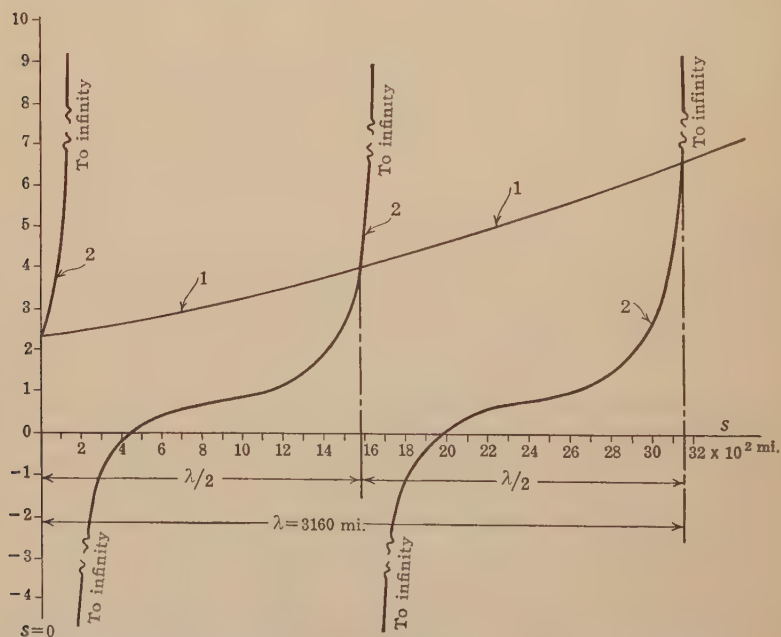


FIG. 6. Graphic determination of the wavelength of forced oscillations in a loaded transmission line.

Fig. 6 represents a plot of Eq. (51) for the line whose characteristics are specified in Art. 7. Curve 1 in this figure represents the term $(E_1/E_2) \epsilon^{2\delta s}$ plotted as ordinate against s as abscissa. Curve 2 represents $\cos(\theta + \gamma) - \cot(\psi s + \theta) \sin(\theta + \gamma)$ plotted as ordinate against s as abscissa. The intersections of the two curves occur at points whose abscissas are 1580 and 3160 miles respectively. The first intersection corresponds to $(\lambda/2)$ and the second represents the wavelength λ . This result checks very closely with the results obtained from Fig. 4.

Strictly speaking, it is not necessary to plot curve 2 for values of $(\psi s + \theta)$ greater than π because the first intersection of curves 1 and 2 represents half a wavelength and the total wavelength may be obtained by multiplying $(\lambda/2)$ by 2. However if a check on the accuracy of this value is required it is best to draw curves 2 for the whole range of $0 \leq (\psi s + \theta) \leq 2\pi$. The work can be greatly simplified by computing only such values of the functions (the right and left parts of Eq. 52) which fall within the range of the points of intersection of the curves 1 and 2 (Fig. 6).

(b₁) *Resonant Frequency in a Circuit Carrying No Load.* — Let a circuit of length L have distributed characteristics R , $\mathcal{L}$, G , and C ; and let it be required to determine the frequency f , at which the line oscillates at no load, that is, such a frequency that Eq. (44b) is satisfied, or:

$$L = q \pi / 2 \psi \quad (52)$$

Since ψ (see Eq. 26e) is a function of f , Eq. (52) can be solved for f , giving more than one frequency at which a given line undergoes forced oscillations. The values of these frequencies depend upon the value of q . But they are all odd multiples of the fundamental (computed for $q = 1$), because q is always an odd integer.

(b₂) *Resonant Frequency in a Circuit Carrying Load.* — The condition for forced oscillations in a loaded circuit of length L is given in Eq. (47), which can be transformed to the form:

$$\begin{aligned} E_1 \epsilon^{\delta L} [\sin \psi L \cos \theta + \cos \psi L \sin \theta] \\ - E_2 \epsilon^{-\delta L} [\sin \psi L \cos \gamma - \cos \psi L \sin \gamma] = 0 \end{aligned} \quad (53)$$

Collecting terms of $\sin \psi L$ and of $\cos \psi L$ we have:

$$\begin{aligned} \sin \psi L [E_1 \epsilon^{\delta L} \cos \theta - E_2 \epsilon^{-\delta L} \cos \gamma] \\ = -\cos \psi L [E_1 \epsilon^{\delta L} \sin \theta + E_2 \epsilon^{-\delta L} \sin \gamma] \end{aligned} \quad (54)$$

whence:

$$\tan \psi L = [E_2 \epsilon^{-\delta L} \sin \gamma + E_1 \epsilon^{\delta L} \sin \theta] / [E_2 \epsilon^{-\delta L} \cos \gamma - E_1 \epsilon^{\delta L} \cos \theta] \quad (55)$$

Eq. (55) gives the value of ψL in radians, from which the value of ψ and consequently the frequency or frequencies of forced oscillations of a transmission circuit may be determined. It is evident that there

exists more than one value of the angle ψL whose tangent satisfies Eq. (55). Hence there is more than one frequency at which a loaded transmission circuit undergoes forced oscillations.

Several problems given at the end of the chapter illustrate the applicability of this treatment to special cases.

11. Power and Efficiency in a Circuit with Distributed Characteristics. — Just as current and voltage are functions of distance, so power in a circuit with distributed characteristics is a function of distance. The transmission engineer is, however, generally interested in the power supplied at the generating station and the power received at the load end of the line. The power expended in the line is designated as "line loss" and found by actually subtracting the receiver power from the generator power.

Fig. 2 gives the values of the vectors $\mathbf{E}_g$ and $\mathbf{I}_g$ at any assumed K.V.A. and P.f. at the receiver end. The power supplied by the generator can be easily obtained by simply taking the product:

$$P = \mathbf{E}_g \bar{\mathbf{I}}_g \quad (56)$$

The real part of this product is the average power delivered by the generator; the imaginary part is the reactive power; and the modulus of the quantity is the K.V.A. supplied by the generator.

The loss due to transmission is:

$$P_L = P_g - P_r \quad (57)$$

where P_g is the average power supplied by the generator end and P_r is the average power at the receiver end.

The percentage of efficiency of the line is:

$$\text{Eff.} = (P_r/P_g) \times 100 \text{ per cent} \quad (58)$$

PROBLEMS

1. Name circuits, other than power transmission lines, whose characteristics must be treated as distributed characteristics.

2. Consult books and periodicals which deal with the computations of the distributed characteristics of circuits.*

3a. Eqs. (7) and (10) are the general differential equations for a circuit with distributed characteristics. In the treatment given in Art. 3, it was assumed that

* The following texts discuss these topics:

(a) Electric Power Transmission and Distribution, by L. F. Woodruff.

(b) Electrical Characteristics of Transmission Circuits, by Wm. Nesbit (a Westinghouse publication).

(c) The Electric Circuit and the Magnetic Circuit, by V. Karapetoff.

(d) Alternating Currents, by Alexander Russell.

e and i are sine functions of time. Consider that e and i are continuous, that is, are independent of time. Show that the solutions of these equations for the steady state are:

$$\left. \begin{aligned} E &= E_r \cosh (s \sqrt{RG}) + I_r (\sqrt{R/G}) \sinh (s \sqrt{RG}) & (a) \\ I &= I_r \cosh (s \sqrt{RG}) + E_r (\sqrt{R/G}) \sinh (s \sqrt{RG}) & (b) \end{aligned} \right\} \quad (59)$$

or:

$$\left. \begin{aligned} E &= E_g \cosh (s' \sqrt{RG}) + I_g (\sqrt{R/G}) \sinh (s' \sqrt{RG}) & (a) \\ I &= I_g \cosh (s' \sqrt{RG}) + E_g (\sqrt{R/G}) \sinh (s' \sqrt{RG}) & (b) \end{aligned} \right\} \quad (60)$$

where

s and s' are distances measured from the receiver and generator end respectively.
 R and G are the resistance and the leakage conductance of the line per unit length.

E_r and I_r are the Emf. and the current at the receiver end.

E_g and I_g are the Emf. and the current at the sending end.

3b. By definition:

$$\left. \begin{aligned} \cosh \bar{K}s &= (\epsilon^{\bar{K}s} + \epsilon^{-\bar{K}s})/2 & (a) \\ \sinh \bar{K}s &= (\epsilon^{\bar{K}s} - \epsilon^{-\bar{K}s})/2 & (b) \end{aligned} \right\} \quad (61)$$

Show that when $\bar{K}$ is complex such that:

$$\bar{K} = \delta + j\psi \quad (62)$$

then:

$$\left. \begin{aligned} \cosh \bar{K}s &= \cosh \delta s \cos \psi s + j \sinh \delta s \sin \psi s & (a) \\ \sinh \bar{K}s &= \sinh \delta s \cos \psi s + j \cosh \delta s \sin \psi s & (b) \end{aligned} \right\} \quad (63)^*$$

Solution for Eq. (63a). — Substituting Eq. (62) in (61) we have:

$$\begin{aligned} \cosh \bar{K}s &= [\epsilon^{\delta s} \epsilon^{j\psi s} + \epsilon^{-\delta s} \epsilon^{-j\psi s}]/2 \\ &= [\epsilon^{\delta s} (\cos \psi s + j \sin \psi s) + \epsilon^{-\delta s} (\cos \psi s - j \sin \psi s)]/2 \\ &= 0.5 (\epsilon^{\delta s} + \epsilon^{-\delta s}) \cos \psi s + j 0.5 (\epsilon^{\delta s} - \epsilon^{-\delta s}) \sin \psi s \\ &= \cosh \delta s \cos \psi s + j \sinh \delta s \sin \psi s \end{aligned}$$

The proof of Eq. (63b) is left as an exercise to the reader.

3c. Show that when $\bar{K}$ in Eq. (62) is imaginary, that is, when $\delta = 0$, then:

$$\left. \begin{aligned} \cosh \bar{K}s &= \cosh j\psi s = \cos \psi s & (a) \\ \sinh \bar{K}s &= \sinh j\psi s = j \sin \psi s & (b) \end{aligned} \right\} \quad (64)$$

3d. A 220-K.v., 60-cycle line is 400 miles long and supplies a 100,000-K.V.A., 80 per cent P.f. lagging load. The characteristics of the line per mile are:

$$R = 0.114 \text{ ohm}, \quad X_L = 0.778 \text{ ohm}, \quad G = 0, \quad B_c = 5.58 \times 10^{-6} \text{ mho.}$$

Compute the vector sender current and voltage.

Solution. —

$$\begin{aligned} Z &= R + jX_L = 0.114 + j0.778 \\ Y &= G + jB_0 = 0 + j5.58 \times 10^{-6} \\ \bar{K} &= \sqrt{ZY} = 2.09 \times 10^{-3} \angle 85^\circ 50' \\ Z_0 &= \sqrt{Z/Y} = 375 \angle -4^\circ 10' \\ \bar{Y}_0 &= 1/Z_0 = 2.665 \times 10^{-3} \angle 4^\circ 10' \\ \bar{K}L &= 2.09 \times 10^{-3} \times 400 \angle 85^\circ 50' = 0.061 + j0.835 = \delta L + j\psi L \end{aligned} \quad \text{See Eqs. (26)}$$

* The tables given in Appendix III are for the real values of the argument x . Hence Eqs. (63) should be used in actually computing $\cosh \bar{K}s$ and $\sinh \bar{K}s$.

Using Eqs. (56) we have:

$$\begin{aligned}\cosh \bar{K}L &= 0.672 + j0.0452 = 0.673 \angle 3^\circ 50' \\ \sinh \bar{K}L &= 0.041 + j0.743 = 0.744 \angle 86^\circ 50'\end{aligned}$$

$$\begin{aligned}I_r &= 100,000/220 \sqrt{3} = 262 \text{ amp.} \\ E_r &= 220/\sqrt{3} = 127 \text{ K.v. to neutral.}\end{aligned}$$

Taking E_r along the reference axis we have:

$$\begin{aligned}E_r &= 127,000 + j0 = 127,000 \angle 0^\circ \\ I_r &= 262 \angle -36^\circ 50'\end{aligned}$$

Using Eqs. (25) we have:

$$\begin{aligned}E_g &= (127,000 \angle 0^\circ)(0.673 \angle 3^\circ 50') + (262 \angle -36^\circ 50')(375 \angle -4^\circ 10')(0.744 \angle 86^\circ 50') \\ &= 85,400 \angle 3^\circ 50' + 73,000 \angle 45^\circ 50' = 136,000 + j58,000 \\ I_g &= (262 \angle -36^\circ 50')(0.673 \angle 3^\circ 50') + (127 \times 10^3 \angle 0^\circ)(2.665 \times 10^{-3} \angle 4^\circ 10') \\ &\quad (0.744 \angle 86^\circ 50') = 176 \angle -33^\circ + 252 \angle 91^\circ = 141 + j153\end{aligned}$$

3e. A long transmission line is to be designed to operate at a receiver voltage of 220 K.v. (between lines), a frequency of 60 cycles per second, a length of 300 miles, and a load of 200,000 K.V.A. at a lagging power factor of 85 per cent. The designing engineer has made the following specifications and turned them over to you for further computations:

- (a) Number of circuits 2.
- (b) Net area of conductor 795,000 c.m.
- (c) Diameter of conductor 1,093 in.
- (d) Type of conductor A.C.S.R.
- (e) Spacing 21 ft.
- (f) Arrangement of circuits $\begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix}$
- (g) Type of insulators, suspension.
- (h) Number of insulators per string, 13.
- (i) Towers per mile, nine.

You are asked to determine the line characteristics $\bar{Z}$, $\bar{Y}$, $\bar{K}$, $\bar{Z}_0$, and $\bar{Y}_0$, assuming that the leakage along the line is zero, and to compute the generator voltage and current assuming that E_r is taken along the reference axis.

4. The characteristics of the line for which the current and voltage loci are given in Fig. 2 are: $\bar{Z} = 0.114 + j0.792$ ohm per mile and $\bar{Y} = 0 + j5.49 \times 10^{-6}$ mho per mile. The receiver Emf. to neutral is 127 K.v. and the full load K.V.A. is 200,000. Check at least one point on the Emf. and current loci.

5a. The line specified in problem 3e is to operate at constant generator and receiver voltage and the generator voltage is to be 220 K.v. between lines. Assume that a synchronous condenser, installed at the receiver, operates at 5 per cent leading or lagging P.f. Find (a) the vector value of the synchronous condenser current, (b) the vector value of the generator voltage, and (c) the vector value of the generator current.

5b. Draw a figure, on large cross-section paper, similar to Fig. 2, and check the results obtained in problem 5a.

6. Using the figure obtained in problem 5b, determine the minimum K.V.A. synchronous condenser and the generator voltage E'_g which would give constant voltage operation at full load and at no load for the line specified in problem 3e.

7a. Read from Figs. 3 the values of the vector voltage and current at intervals of 200 miles from the receiver end.

7b. Draw figures similar to Figs. 3 for the line specified in problem 3e.

7c. What modifications would Figs. 3 undergo when the resistance and leakance of the line are negligible?

7d. Draw figures corresponding to Figs. 3 for the line specified in problem 3e, assuming that $R = 0$.

8. From the figure obtained in problem 7d draw a figure similar to Fig. 4 and determine the wavelength of the space waves.

9. Draw an analogy similar to that described in Art. 9 between the space waves of a transmission line and the space waves of air in a wind musical instrument.

10a. Consider that the line specified in problem 3e is unloaded. Compute the wavelength λ of the space waves:

(a) Neglecting the resistance.

(b) Taking the resistance into consideration.

10b. Let the line specified in problem 3e be loaded. What would be the wavelength:

(a) When the resistance is negligible?

(b) When the resistance is not negligible?

10c. At what frequencies will the line specified in problem 3e undergo forced oscillations:

(a) When its resistance is neglected?

(b) When its resistance is not neglected?

11a. What is the power supplied by the generator in problem 3e?

11b. Compute the power supplied by the generator in problem 5a.

11c. What are the efficiencies of the lines in problems 11a and 11b?

APPENDIX I

THEOREMS USED IN THIS BOOK AND THEIR PROOFS

Theorem I. — The sum of ($n \geq 2$) coplanar vectors, of equal moduli and equally displaced from each other by an angle $\alpha = 2\pi/n$ degrees, is zero.

Proof. — The complex expressions for the various vectors are:

$$\left. \begin{aligned} A_1 &= A\epsilon^{j\alpha} & (a) \\ A_2 &= A\epsilon^{j^2\alpha} & (b) \\ \vdots & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ A_n &= A\epsilon^{jn\alpha} & (n) \end{aligned} \right\} \quad (1.1)$$

Adding equations (1.1) we have:

$$A_1 + A_2 + \dots + A_n = A(\epsilon^{j\alpha} + \epsilon^{j^2\alpha} + \dots + \epsilon^{jn\alpha}) \quad (1.2)$$

If it can be shown that $(\epsilon^{j\alpha} + \epsilon^{j^2\alpha} + \dots + \epsilon^{jn\alpha}) = 0$, then the problem is solved. To show that such is the case let:

$$B = \epsilon^{j\alpha} \quad (1.3)$$

$$\text{Then} \quad B^n = \epsilon^{jn\alpha} = \epsilon^{jn(2\pi/n)} = \epsilon^{j^2\pi} = 1 \quad (1.4)$$

$$\text{Hence:} \quad B^n - 1 = 0 \quad (1.5)$$

Factoring (1.5) we have:

$$(B - 1)(B^{n-1} + B^{n-2} + B^{n-3} + \dots + B + 1) = 0 \quad (1.6)$$

But $(B - 1) \neq 0$.

Therefore

$$\left. \begin{aligned} (B^{n-1} + B^{n-2} + \dots + B + 1) &= 0 \\ (\epsilon^{j\alpha} + \epsilon^{j^2\alpha} + \dots + \epsilon^{jn\alpha}) &= 0 \end{aligned} \right\} \quad \text{Q.E.D.} \quad (1.7)$$

This proof illustrates the utility of complex numbers as a mathematical tool. Indeed the proof of this theorem by geometrical means is much more involved than the one given here.

Theorem II. — If $s \neq n$, then

$$U = \int_0^{2\pi} \sum_{n=1}^{n=s-1} + \sum_{n=s+1}^{n=\infty} C_n \sin n(x + \alpha_n) \sin s(x + \alpha_s) dx = 0 \quad (2.1)$$

Proof. — Using the trigonometric transformation,

$$\sin a \sin b = 0.5 [\cos (a - b) - \cos (a + b)]$$

we have:

$$\begin{aligned} U = \int_0^{2\pi} \left[\sum_{n=1}^{n=s-1} + \sum_{n=s+1}^{n=\infty} 0.5 C_n \left\{ \cos [(n - s)x + n\alpha_n - s\alpha_s] \right. \right. \\ \left. \left. - \cos [(n + s)x + n\alpha_n + s\alpha_s] \right\} \right] dx \end{aligned} \quad (2.2)$$

Integrating and substituting limits we obtain:

$$U = \left. \begin{aligned} & \sum_{n=1}^{n=s-1} + \sum_{n=s+1}^{n=\infty} \left[0.5 C_n / (n-s) \right] \left\{ \sin (s\alpha_s - n\alpha_n) - \sin (s\alpha_s - n\alpha_n) \right\} \\ & + \sum_{n=1}^{n=s-1} + \sum_{n=s+1}^{n=\infty} \left[0.5 C_n / (n-s) \right] \left\{ \sin (s\alpha_s + n\alpha_n) - \sin (s\alpha_s + n\alpha_n) \right\} \end{aligned} \right\} \quad (2.3)$$

$$\begin{aligned} \text{But} & \sin (s\alpha_s - n\alpha_n) - \sin (s\alpha_s - n\alpha_n) \equiv 0 \\ \text{and} & \sin (s\alpha_s + n\alpha_n) - \sin (s\alpha_s + n\alpha_n) \equiv 0 \end{aligned} \quad (2.4)$$

Hence each term in the summation is identically equal to zero and

$$U \equiv 0 \quad \text{Q.E.D.} \quad (2.5)$$

Theorem III. — If Fourier's series is true, then the coefficients of the cosine and sine harmonics are:

$$\left. \begin{aligned} A_s &= (1/\pi) \int_0^{2\pi} f(x) \cos sx \, dx \quad (a) \\ B_s &= (1/\pi) \int_0^{2\pi} f(x) \sin sx \, dx \quad (b) \end{aligned} \right\} \quad (3.1)$$

Proof. — Let the sine and cosine terms of any harmonic in Fourier's series be combined into one term with an initial angle α_n (see Art. 8, Chapter XIII).

We have:

$$f(x) = A_0 + C_s \sin s(x + \alpha_s) + \sum_{n=1}^{n=s-1} + \sum_{n=s+1}^{n=\infty} C_n \sin (x + \alpha_n) \quad (3.2)$$

Multiplying both sides of Eq. (3.2) by $\sin s(x + \alpha_s) dx$ and integrating from 0 to 2π we obtain:

$$\int_0^{2\pi} f(x) \sin s(x + \alpha_s) dx = A_0 \int_0^{2\pi} \sin s(x + \alpha_s) dx + \int_0^{2\pi} C_s \sin^2 s(x + \alpha_s) dx + U \quad (3.3)$$

where U is expressed by Eq. (2.1) and is proved to be equal to zero.

Using the transformation $\sin^2 a = 0.5 - 0.5 \cos 2a$, integrating the right-hand side of Eq. (3.3), substituting limits, and solving for C_s we have:

$$\left. \begin{aligned} C_s &= (1/\pi) \int_0^{2\pi} f(x) \sin s(x + \alpha_s) dx \quad (a) \\ &= (1/\pi) \int_0^{2\pi} f(x) [\sin sx \cos s\alpha_s + \cos sx \sin s\alpha_s] dx \quad (b) \\ &= (1/\pi) \cos s\alpha_s \int_0^{2\pi} f(x) \sin sx \, dx + (1/\pi) \sin s\alpha_s \int_0^{2\pi} f(x) \cos sx \, dx \quad (c) \end{aligned} \right\} \quad (3.4)$$

Expanding Eq. (78c) of Chapter XIII, we have:

$$A_s \cos sx + B_s \sin sx = C_s \sin sx \cos s\alpha_s + C_s \cos sx \sin s\alpha_s \quad (3.5)$$

If Eq. (3.5) is to hold true for all values of sx it must be an identity; in other words, the coefficients of $\sin sx$ and of $\cos sx$ must be equal, or:

$$\left. \begin{aligned} A_s &= C_s \sin s\alpha_s \quad (a) \\ B_s &= C_s \cos s\alpha_s \quad (b) \end{aligned} \right\} \quad (3.6)$$

Multiplying Eq. (3.6a) by $\sin s\alpha_s$ and Eq. (3.6b) by $\cos s\alpha_s$, and adding we obtain:

$$C_s = A_s \sin s\alpha_s + B_s \cos s\alpha_s \quad (3.7)$$

Comparing Eq. (3.7) with Eq. (3.4c) we conclude that:

$$\left. \begin{aligned} A_s &= (1/\pi) \int_0^{2\pi} f(x) \cos sx \, dx \quad (a) \\ B_s &= 1/\pi \int_0^{2\pi} f(x) \sin sx \, dx \quad (b) \end{aligned} \right\} \quad (3.8)$$

Theorem IV. — If E_n and I_n are positive numbers, then:

$$\left. \begin{aligned} &\left[\sum E_n I_n / \sqrt{\sum E_n^2 \sum I_n^2} \right] \leq 1 \quad (a) \\ \text{or} \quad &\left[(\sum E_n I_n)^2 / \sum E_n^2 \sum I_n^2 \right] \leq 1 \quad (b) \end{aligned} \right\} \quad (4.1)$$

Proof. — We shall prove the inequality for just four terms. The extension of the proof to n terms then becomes apparent.

Write the denominator of Eq. (4.1b) in the form:

$$\begin{aligned} (E_1^2 + E_2^2 + E_3^2 + E_4^2) (I_1^2 + I_2^2 + I_3^2 + I_4^2) &= (E_1 I_1 + E_2 I_2 + E_3 I_3 + E_4 I_4)^2 \\ &\quad + \left| \frac{E_1}{I_1} \frac{E_2}{I_2} \right|^2 + \left| \frac{E_1}{I_1} \frac{E_3}{I_3} \right|^2 + \left| \frac{E_1}{I_1} \frac{E_4}{I_4} \right|^2 \\ &\quad + \left| \frac{E_2}{I_2} \frac{E_3}{I_3} \right|^2 + \left| \frac{E_2}{I_2} \frac{E_4}{I_4} \right|^2 + \left| \frac{E_3}{I_3} \frac{E_4}{I_4} \right|^2 \end{aligned} \quad (4.2)$$

But the numerator of Eq. (4.1b) is:

$$(E_1 I_1 + E_2 I_2 + E_3 I_3 + E_4 I_4)^2 \quad (4.3)$$

Comparing Eq. (4.3) and the right-hand side of Eq. (4.2) we find that the latter exceeds the former by the sum of the squares of all the determinants.

We thus conclude that:

- (1) If the sum of the squares of all the determinants in Eq. (4.2) is greater than zero, then the fraction is less than 1.
- (2) If the sum of the squares of all the determinants in Eq. (4.2) is equal to zero, then the fraction is equal to 1.

Thus the theorem is proved.

Condition (2) obtains when:

$$(E_1/I_1) = (E_2/I_2) = (E_3/I_3) = (E_4/I_4) \quad (4.4)$$

This condition physically means that the resistance of the circuit does not change with the order of the harmonic or that the skin effect is negligible.

Theorem V. — The general solution of a linear differential equation of the first order with constant coefficients having the form:

$$(dx/dt) + ax = by \quad (5.1)$$

where x and y are functions of t ; a and b are constants, is:

$$x = e^{-at} \int by e^{at} dt + K e^{-at} \quad (5.2)$$

Proof. — The result of multiplying Eq. (5.1) by e^{at} is:

$$(dx/dt) e^{at} + ax e^{at} = by e^{at} \quad (5.3)$$

Let:

$$s = x e^{at} \quad (5.4)$$

$$ds/dt = (dx/dt) e^{at} + ax e^{at} \quad (5.5)$$

But the right-hand side of Eq. (5.5) is identical with the left-hand side of Eq. (5.3). Therefore:

$$ds/dt = by \epsilon^{at} \quad (5.6)$$

Whence:

$$s = \int by \epsilon^{at} dt + K \quad (5.7)$$

Substituting in Eq. (5.7) the value of s from Eq. (5.4) we have:

$$x = \epsilon^{-at} \int by \epsilon^{at} dt + K \epsilon^{-at} \quad \text{Q.E.D.} \quad (5.8)$$

Theorem VI. — The general solution of a linear differential equation of the 2nd order with constant coefficients such as

$$(d^2x/dt^2) + a(dx/dt) + bx = hy \quad (6.1)$$

where x and y are functions of t and a, b, h are constants, is of the form:

$$x = \epsilon^{-vt} \int \left[\epsilon^{-ut} \int hy \epsilon^{ut} dt + K_1 \epsilon^{-ut} \right] \epsilon^{vt} dt + K_2 \epsilon^{-vt} \quad (6.2)$$

where u and v are expressed by Eq. (6.7) below.

Proof. — Let:

$$(d/dt) = D \quad (6.3)$$

Then Eq. (6.1) may be written in the form:

$$D^2x + aDx + bx = hy \quad (6.4)$$

Factoring x in Eq. (6.4) we have:

$$(D^2 + aD + b)x = hy \quad (6.5)$$

If the expression within parenthesis is factored as an algebraic expression in D we obtain:

$$(D + u)(D + v)x = hy \quad (6.6)$$

where:

$$\left. \begin{aligned} u &= (a + \sqrt{a^2 - 4b})/2 & (a) \\ v &= (a - \sqrt{a^2 - 4b})/2 & (b) \\ u + v &= a & (c) \\ uv &= b & (d) \end{aligned} \right\} \quad (6.7)$$

We maintain that the left-hand sides of Eq. (6.6) and (6.1) are identical provided the algebraic quantities $(D + u)(D + v)$ are used as operators on x .

To prove this statement let the operators in Eq. (6.6) act on x . We have:

$$\left. \begin{aligned} (D + v)x &= dx/dt + vx & (a) \\ (D + u)[(dx/dt) + vx] &= (d^2x/dt^2) + v(dx/dt) + u(dx/dt) + uvx & (b) \\ &= (d^2x/dt^2) + (u + v)(dx/dt) + uvx & (c) \\ &= (d^2x/dt^2) + a(dx/dt) + bx & \text{Q.E.D.} \quad (d) \end{aligned} \right\} \quad (6.8)$$

Having reduced Eq. (6.1) to the symbolic form, the solution becomes very similar to that submitted in Theorem V. Thus let:

$$(D + v)x = s \quad (6.9)$$

Substituting Eq. (6.9) in Eq. (6.6) we have:

$$(D + u)s = hy \quad (6.10)$$

$$(ds/dt) + us = hy \quad (6.11)$$

But the general solution of Eq. (6.11) is (according to Eq. (5.2)):

$$s = \epsilon^{-ut} \int hy \epsilon^{ut} dt + K_1 \epsilon^{-ut} \quad (6.12)$$

Now let $(D + v)$ in Eq. (6.9) operate on x . This gives:

$$(dx/dt) + vx = s \quad (6.13)$$

The solution of Eq. (6.13) is given by Eq. (5.2). This is:

$$x = \epsilon^{-vt} \int s \epsilon^{vt} dt + K_2 \epsilon^{-vt} \quad (6.14)$$

Substituting in Eq. (6.14) the value of s from Eq. (6.12) we have:

$$x = \epsilon^{-vt} \int \left[\epsilon^{-ut} \int hy \epsilon^{ut} dt + K_1 \epsilon^{-ut} \right] \epsilon^{vt} dt + K_2 \epsilon^{-vt} \quad \text{Q.E.D.} \quad (6.15)$$

APPENDIX II

MATHEMATICAL AND PHYSICAL TABLES

The tables contained in this appendix are included for the convenience of the users of this text. Following is a list of the tables:

<i>Table Number</i>	<i>Page</i>
I. Circular or Trigonometric Functions	350
II. Hyperbolic and Exponential Functions	352
III. Resistance of Stranded Copper Conductors	358
IV. Resistance of A.C.S.R. Conductors	359
V. 60-Cycle Inductive Reactance of Copper Conductors	360
VI. 60-Cycle Inductive Reactance of A.C.S.R. Conductors	361
VII. 60-Cycle Capacitive Susceptance of Copper Conductors	362
VIII. 60-Cycle Capacitive Susceptance of A.C.S.R. Conductors	363
IX to XV. Tables for Evaluating the Constants A_n and B_n in a Fourier's Series	364

NOTE. — Tables I and II were abstracted from the Smithsonian Mathematical Tables.

Tables III to VIII inclusive were borrowed from "Electrical Characteristics of Transmission Circuits" by kind permission of the Westinghouse Electric and Manufacturing Co.

Tables IX to XV were prepared by the author.

TABLE I
CIRCULAR OR TRIGONOMETRIC FUNCTIONS

x Rad.	x Degrees	$\sin x$	$\cos x$	$\tan x$	x Rad.	x Degrees	$\sin x$	$\cos x$	$\tan x$
0.00	00°00.0'	0.00000	1.00000	0.00000	0.40	22°55.1'	0.38942	0.92106	0.42279
0.01	00°34.4'	0.01000	0.99995	0.01000	0.41	23°29.5'	0.39861	0.91712	0.43463
0.02	01°08.7'	0.02000	0.99980	0.02000	0.42	24°03.8'	0.40776	0.91309	0.44657
0.03	01°43.2'	0.03000	0.99955	0.03001	0.43	24°38.2'	0.41687	0.90897	0.45862
0.04	02°17.5'	0.03999	0.99920	0.04002	0.44	25°12.7'	0.42594	0.90475	0.47078
0.05	02°51.9'	0.04998	0.99875	0.05004	0.45	25°47.0'	0.43497	0.90045	0.48306
0.06	03°26.3'	0.05996	0.99820	0.06007	0.46	26°21.4'	0.44395	0.89605	0.49545
0.07	04°00.7'	0.06994	0.99755	0.07011	0.47	26°55.7'	0.45289	0.89157	0.50797
0.08	04°35.0'	0.07991	0.99680	0.08017	0.48	27°30.1'	0.46178	0.88699	0.52061
0.09	05°09.4'	0.08988	0.99595	0.09024	0.49	28°04.5'	0.47063	0.88233	0.53339
0.10	05°43.8'	0.09983	0.99500	0.10033	0.50	28°38.9'	0.47943	0.87758	0.54630
0.11	06°18.1'	0.10978	0.99396	0.11045	0.51	29°13.3'	0.48818	0.87274	0.55936
0.12	06°52.5'	0.11971	0.99281	0.12058	0.52	29°47.7'	0.49688	0.86782	0.57256
0.13	07°26.9'	0.12963	0.99156	0.13074	0.53	30°22.0'	0.50553	0.86281	0.58592
0.14	08°01.3'	0.13954	0.99022	0.14092	0.54	30°56.4'	0.51414	0.85771	0.59943
0.15	08°35.7'	0.14944	0.98877	0.15114	0.55	31°30.8'	0.52269	0.85252	0.61311
0.16	09°10.0'	0.15932	0.98723	0.16138	0.56	32°05.1'	0.53119	0.84726	0.62695
0.17	09°44.4'	0.16918	0.98558	0.17166	0.57	32°39.5'	0.53963	0.84190	0.64097
0.18	10°18.8'	0.17903	0.98384	0.18197	0.58	33°13.9'	0.54802	0.83646	0.65517
0.19	10°53.2'	0.18886	0.98200	0.19232	0.59	33°48.3'	0.55636	0.83094	0.66956
0.20	11°27.5'	0.19867	0.98007	0.20271	0.60	34°22.7'	0.56464	0.82534	0.68414
0.21	12°01.9'	0.20846	0.97803	0.21314	0.61	34°57.0'	0.57287	0.81965	0.69892
0.22	12°36.3'	0.21823	0.97590	0.22362	0.62	35°31.4'	0.58104	0.81388	0.71391
0.23	13°10.7'	0.22798	0.97367	0.23414	0.63	36°05.8'	0.58914	0.80803	0.72911
0.24	13°45.1'	0.23770	0.97134	0.24472	0.64	36°40.1'	0.59720	0.80210	0.74454
0.25	14°19.4'	0.24740	0.96891	0.25534	0.65	37°14.5'	0.60519	0.79608	0.76020
0.26	14°53.8'	0.25708	0.96639	0.26602	0.66	37°48.9'	0.61312	0.78999	0.77610
0.27	15°28.2'	0.26673	0.96377	0.27676	0.67	38°23.3'	0.62099	0.78382	0.79225
0.28	16°02.6'	0.27636	0.96106	0.28755	0.68	38°57.7'	0.62879	0.77757	0.80866
0.29	16°36.9'	0.28595	0.95824	0.29841	0.69	39°32.0'	0.63654	0.77125	0.82534
0.30	17°11.3'	0.29552	0.95534	0.30934	0.70	40°06.4'	0.64422	0.76484	0.84229
0.31	17°45.7'	0.30506	0.95233	0.32033	0.71	40°40.8'	0.65183	0.75836	0.85953
0.32	18°20.1'	0.31457	0.94924	0.33139	0.72	41°15.1'	0.65938	0.75181	0.87707
0.33	18°54.4'	0.32404	0.94604	0.34252	0.73	41°49.6'	0.66687	0.74517	0.89492
0.34	19°28.8'	0.33349	0.94275	0.35374	0.74	42°24.0'	0.67429	0.73847	0.91309
0.35	20°03.2'	0.34290	0.93937	0.36503	0.75	42°58.3'	0.68164	0.73169	0.93160
0.36	20°37.6'	0.35227	0.93590	0.37640	0.76	43°32.7'	0.68892	0.72484	0.95045
0.37	21°11.9'	0.36162	0.93233	0.38786	0.77	44°07.1'	0.69614	0.71791	0.96967
0.38	21°46.3'	0.37092	0.92866	0.39941	0.78	44°41.5'	0.70328	0.71091	0.98926
0.39	22°20.7'	0.38019	0.92491	0.41105	0.79	45°15.8'	0.71035	0.70385	1.00920

TABLE I — (Continued)
CIRCULAR OR TRIGONOMETRIC FUNCTIONS

x Rad.	x Degrees	$\sin x$	$\cos x$	$\tan x$	x Rad.	x Degrees	$\sin x$	$\cos x$	$\tan x$
0.80	45°50.2'	0.71736	0.69671	1.02960	1.20	68°45.3'	0.93204	0.36236	2.57220
0.81	46°24.6'	0.72429	0.68950	1.05050	1.21	69°19.7'	0.93562	0.35302	2.65030
0.82	46°59.0'	0.73115	0.68222	1.07170	1.22	69°54.0'	0.93910	0.34365	2.73280
0.83	47°33.3'	0.73793	0.67488	1.09340	1.23	70°28.4'	0.94249	0.33424	2.81980
0.84	48°07.7'	0.74464	0.66746	1.11560	1.24	71°02.8'	0.94578	0.32480	2.91190
0.85	48°42.1'	0.75128	0.65998	1.13830	1.25	71°37.2'	0.94898	0.31532	3.00960
0.86	49°16.5'	0.75784	0.65244	1.16160	1.26	72°11.6'	0.95209	0.30582	3.11330
0.87	49°50.8'	0.76433	0.64483	1.18530	1.27	72°45.9'	0.95510	0.29628	3.22360
0.88	50°25.2'	0.77074	0.63715	1.20970	1.28	73°20.3'	0.95802	0.28672	3.34130
0.89	50°59.6'	0.77707	0.62941	1.23460	1.29	73°54.7'	0.96084	0.27712	3.46720
0.90	51°34.0'	0.78333	0.62161	1.26020	1.30	74°29.1'	0.96356	0.26750	3.60210
0.91	52°08.3'	0.78950	0.61375	1.28640	1.31	75°03.5'	0.96618	0.25785	3.74710
0.92	52°42.7'	0.79560	0.60582	1.31330	1.32	75°37.8'	0.96872	0.24818	3.90330
0.93	53°17.1'	0.80162	0.59783	1.34090	1.33	76°12.2'	0.97115	0.23848	4.07230
0.94	53°51.5'	0.80756	0.58979	1.36920	1.34	76°46.6'	0.97348	0.22875	4.25560
0.95	54°25.9'	0.81342	0.58168	1.39840	1.35	77°20.9'	0.97572	0.21901	4.45520
0.96	55°00.2'	0.81919	0.57352	1.42840	1.36	77°55.3'	0.97786	0.20924	4.67340
0.97	55°34.6'	0.82489	0.56530	1.45920	1.37	78°29.7'	0.97991	0.19945	4.91310
0.98	56°09.0'	0.83050	0.55702	1.49100	1.38	79°04.1'	0.98185	0.18964	5.17740
0.99	56°43.4'	0.83603	0.54869	1.52370	1.39	79°38.5'	0.98370	0.17981	5.47070
1.00	57°17.8'	0.84147	0.54030	1.55740	1.40	80°12.9'	0.98545	0.16997	5.7979
1.01	57°52.1'	0.84683	0.53186	1.59220	1.41	80°47.2'	0.98710	0.16010	6.1654
1.02	58°26.5'	0.85211	0.52337	1.62810	1.42	81°21.6'	0.98865	0.15023	6.5811
1.03	59°00.9'	0.85730	0.51482	1.66520	1.43	81°56.0'	0.99010	0.14033	7.0555
1.04	59°35.3'	0.86240	0.50622	1.70360	1.44	82°30.4'	0.99146	0.13042	7.6018
1.05	60°09.7'	0.86742	0.49757	1.74330	1.45	83°04.7'	0.99271	0.12050	8.2381
1.06	60°44.0'	0.87236	0.48887	1.78440	1.46	83°39.1'	0.99387	0.11057	8.9886
1.07	61°18.4'	0.87720	0.48012	1.82700	1.47	84°13.5'	0.99492	0.10063	9.8874
1.08	61°52.8'	0.88196	0.47133	1.87120	1.48	84°47.9'	0.99588	0.09067	10.9830
1.09	62°27.1'	0.88663	0.46249	1.91710	1.49	85°22.3'	0.99674	0.08071	12.350
1.10	63°01.5'	0.89121	0.45360	1.96480	1.50	85°56.6'	0.99749	0.07074	14.101
1.11	63°35.9'	0.89570	0.44466	2.01430	1.51	86°31.0'	0.99815	0.06076	16.428
1.12	64°10.3'	0.90010	0.43568	2.06600	1.52	87°05.4'	0.99871	0.05077	19.670
1.13	64°44.7'	0.90441	0.42666	2.11980	1.53	87°39.8'	0.99917	0.04079	24.498
1.14	65°19.0'	0.90863	0.41759	2.17590	1.54	88°14.2'	0.99953	0.03079	32.461
1.15	65°53.4'	0.91276	0.40849	2.23450	1.55	88°48.5'	0.99978	0.02079	48.078
1.16	66°27.8'	0.91680	0.39934	2.29580	1.56	89°22.9'	0.99994	0.01080	92.621
1.17	67°02.2'	0.92075	0.39015	2.36000	1.57	89°57.3'	1.00000	0.00080	1255.80
1.18	67°36.5'	0.92461	0.38092	2.42730	$\pi/2$	90°00.0'	1.00000	0.00000	∞
1.19	68°10.9'	0.92837	0.37166	2.49790					

TABLE II
HYPERBOLIC AND EXPONENTIAL FUNCTIONS

z	$\sinh z$	$\cosh z$	e^z	e^{-z}	z	$\sinh z$	$\cosh z$	e^z	e^{-z}
0.000	0.00000	1.00000	1.00000	1.00000	0.200	0.20134	1.02007	1.22140	0.81873
0.005	0.00500	1.00001	1.00501	0.99501	0.205	0.20644	1.02109	1.22753	0.81465
0.010	0.01000	1.00005	1.01005	0.99005	0.210	0.21155	1.02213	1.23368	0.81059
0.015	0.01500	1.00011	1.01511	0.98511	0.215	0.21666	1.02320	1.23986	0.80654
0.020	0.02000	1.00020	1.02020	0.98020	0.220	0.22178	1.02430	1.24608	0.80252
0.025	0.02500	1.00031	1.02532	0.97531	0.225	0.22690	1.02542	1.25232	0.79852
0.030	0.03000	1.00045	1.03046	0.97045	0.230	0.23203	1.02657	1.25860	0.79453
0.035	0.03501	1.00061	1.03562	0.96561	0.235	0.23717	1.02774	1.26491	0.79057
0.040	0.04001	1.00080	1.04081	0.96079	0.240	0.24231	1.02894	1.27125	0.78663
0.045	0.04502	1.00101	1.04603	0.95600	0.245	0.24746	1.03016	1.27762	0.78271
0.050	0.05002	1.00125	1.05127	0.95123	0.250	0.25261	1.03141	1.28403	0.77880
0.055	0.05503	1.00151	1.05654	0.94649	0.255	0.25777	1.03269	1.29046	0.77492
0.060	0.06004	1.00180	1.06184	0.94177	0.260	0.26294	1.03399	1.29693	0.77105
0.065	0.06505	1.00211	1.06716	0.93707	0.265	0.26811	1.03532	1.30343	0.76721
0.070	0.07006	1.00245	1.07251	0.93239	0.270	0.27329	1.03667	1.30996	0.76338
0.075	0.07507	1.00281	1.07788	0.92774	0.275	0.27848	1.03805	1.31653	0.75957
0.080	0.08009	1.00320	1.08329	0.92312	0.280	0.28367	1.03946	1.32313	0.75578
0.085	0.08510	1.00361	1.08872	0.91851	0.285	0.28887	1.04089	1.32976	0.75202
0.090	0.09012	1.00405	1.09417	0.91393	0.290	0.29408	1.04235	1.33642	0.74826
0.095	0.09514	1.00452	1.09966	0.90937	0.295	0.29930	1.04383	1.34313	0.74453
0.100	0.10017	1.00500	1.10517	0.90484	0.300	0.30452	1.04534	1.34986	0.74082
0.105	0.10519	1.00552	1.11071	0.90033	0.305	0.30975	1.04687	1.35663	0.73712
0.110	0.11022	1.00606	1.11628	0.89584	0.310	0.31499	1.04844	1.36342	0.73345
0.115	0.11525	1.00662	1.12188	0.89137	0.315	0.32024	1.05002	1.37026	0.72979
0.120	0.12029	1.00721	1.12750	0.88692	0.320	0.32549	1.05164	1.37713	0.72615
0.125	0.12533	1.00782	1.13315	0.88250	0.325	0.33075	1.05328	1.38403	0.72253
0.130	0.13037	1.00846	1.13883	0.87810	0.330	0.33602	1.05495	1.39097	0.71893
0.135	0.13541	1.00913	1.14454	0.87372	0.335	0.34130	1.05664	1.39794	0.71534
0.140	0.14046	1.00982	1.15028	0.86936	0.340	0.34659	1.05836	1.40495	0.71178
0.145	0.14551	1.01053	1.15604	0.86502	0.345	0.35188	1.06011	1.41199	0.70822
0.150	0.15056	1.01127	1.16183	0.86071	0.350	0.35719	1.06186	1.41907	0.70469
0.155	0.15562	1.01204	1.16766	0.85642	0.355	0.36250	1.06368	1.42618	0.70117
0.160	0.16068	1.01283	1.17351	0.85214	0.360	0.36783	1.06550	1.43333	0.69768
0.165	0.16575	1.01364	1.17939	0.84789	0.365	0.37316	1.06736	1.44051	0.69420
0.170	0.17082	1.01448	1.18531	0.84367	0.370	0.37850	1.06923	1.44774	0.69074
0.175	0.17589	1.01535	1.19125	0.83946	0.375	0.38385	1.07114	1.45499	0.68729
0.180	0.18097	1.01624	1.19722	0.83527	0.380	0.38921	1.07307	1.46229	0.68386
0.185	0.18606	1.01716	1.20322	0.83111	0.385	0.39458	1.07503	1.46961	0.68045
0.190	0.19115	1.01810	1.20925	0.82696	0.390	0.39996	1.07702	1.47698	0.67706
0.195	0.19624	1.01907	1.21531	0.82284	0.395	0.40535	1.07903	1.48438	0.67368

TABLE II — (Continued)
HYPERBOLIC AND EXPONENTIAL FUNCTIONS

z	$\sinh z$	$\cosh z$	e^z	e^{-z}	z	$\sinh z$	$\cosh z$	e^z	e^{-z}
0.400	0.41075	1.08107	1.49183	0.67032	0.600	0.63665	1.18547	1.82212	0.54881
0.405	0.41616	1.08314	1.49930	0.66698	0.605	0.64259	1.18866	1.83125	0.54608
0.410	0.42158	1.08523	1.50682	0.66365	0.610	0.64854	1.19189	1.84043	0.54335
0.415	0.42702	1.08736	1.51437	0.66034	0.615	0.65451	1.19515	1.84966	0.54064
0.420	0.43246	1.08950	1.52196	0.65705	0.620	0.66049	1.19844	1.85893	0.53795
0.425	0.43791	1.09168	1.52959	0.65377	0.625	0.66649	1.20175	1.86825	0.53526
0.430	0.44337	1.09388	1.53726	0.65051	0.630	0.67251	1.20510	1.87761	0.53259
0.435	0.44885	1.09611	1.54496	0.64727	0.635	0.67854	1.20848	1.88702	0.52994
0.440	0.45434	1.09837	1.55271	0.64404	0.640	0.68459	1.21189	1.89648	0.52729
0.445	0.45983	1.10066	1.56049	0.64083	0.645	0.69066	1.21532	1.90599	0.52466
0.450	0.46534	1.10297	1.56831	0.63763	0.650	0.69675	1.21879	1.91554	0.52205
0.455	0.47086	1.10531	1.57617	0.63445	0.655	0.70285	1.22229	1.92514	0.51944
0.460	0.47640	1.10768	1.58407	0.63128	0.660	0.70897	1.22582	1.93479	0.51685
0.465	0.48194	1.11007	1.59201	0.62814	0.665	0.71511	1.22938	1.94449	0.51427
0.470	0.48750	1.11250	1.59999	0.62500	0.670	0.72126	1.23297	1.95424	0.51171
0.475	0.49306	1.11495	1.60802	0.62189	0.675	0.72744	1.23659	1.96403	0.50916
0.480	0.49865	1.11743	1.61607	0.61878	0.680	0.73363	1.24025	1.97388	0.50662
0.485	0.50424	1.11994	1.62418	0.61570	0.685	0.73984	1.24393	1.98377	0.50409
0.490	0.50984	1.12247	1.63232	0.61263	0.690	0.74607	1.24765	1.99372	0.50158
0.495	0.51546	1.12503	1.64050	0.60957	0.695	0.75232	1.25139	2.00371	0.49908
0.500	0.52110	1.12763	1.64872	0.60653	0.700	0.75858	1.25517	2.01375	0.49659
0.505	0.52674	1.13025	1.65699	0.60351	0.705	0.76487	1.25898	2.02385	0.49411
0.510	0.53240	1.13289	1.66529	0.60050	0.710	0.77117	1.26282	2.03399	0.49165
0.515	0.53807	1.13557	1.67364	0.59750	0.715	0.77750	1.26669	2.04419	0.48919
0.520	0.54375	1.13827	1.68203	0.59452	0.720	0.78384	1.27059	2.05443	0.48675
0.525	0.54945	1.14101	1.69046	0.59156	0.725	0.79020	1.27453	2.06473	0.48432
0.530	0.55516	1.14377	1.69893	0.58861	0.730	0.79659	1.27849	2.07508	0.48191
0.535	0.56089	1.14656	1.70745	0.58567	0.735	0.80299	1.28249	2.08548	0.47951
0.540	0.56663	1.14938	1.71601	0.58275	0.740	0.80941	1.28652	2.09594	0.47711
0.545	0.57238	1.15223	1.72461	0.57984	0.745	0.81585	1.29059	2.10644	0.47473
0.550	0.57815	1.15510	1.73325	0.57695	0.750	0.82232	1.29468	2.11700	0.47237
0.555	0.58393	1.15801	1.74194	0.57407	0.755	0.82880	1.29881	2.12761	0.47001
0.560	0.58973	1.16094	1.75067	0.57121	0.760	0.83530	1.30297	2.13828	0.46767
0.565	0.59554	1.16390	1.75945	0.56836	0.765	0.84183	1.30716	2.14899	0.46534
0.570	0.60137	1.16690	1.76827	0.56553	0.770	0.84838	1.31139	2.15977	0.46301
0.575	0.60721	1.16992	1.77713	0.56271	0.775	0.85494	1.31565	2.17059	0.46070
0.580	0.61307	1.17297	1.78604	0.55990	0.780	0.86153	1.31994	2.18147	0.45841
0.585	0.61894	1.17605	1.79499	0.55711	0.785	0.86814	1.32426	2.19241	0.45612
0.590	0.62483	1.17916	1.80399	0.55433	0.790	0.87478	1.32862	2.20340	0.45385
0.595	0.63073	1.18230	1.81303	0.55156	0.795	0.88143	1.33301	2.21444	0.45158

TABLE II — (Continued)
HYPERBOLIC AND EXPONENTIAL FUNCTIONS

z	$\sinh z$	$\cosh z$	e^z	e^{-z}	z	$\sinh z$	$\cosh z$	e^z	e^{-z}
0.800	0.88811	1.33743	2.22554	0.44933	1.000	1.17520	1.54308	2.71828	0.36788
0.805	0.89480	1.34189	2.23670	0.44709	1.010	1.19069	1.55491	2.74560	0.36422
0.810	0.90152	1.34638	2.24791	0.44486	1.020	1.20630	1.56689	2.77320	0.36060
0.815	0.90827	1.35091	2.25918	0.44264	1.030	1.22203	1.57904	2.80107	0.35701
0.820	0.91503	1.35547	2.27050	0.44043	1.040	1.23788	1.59134	2.82922	0.35345
0.825	0.92182	1.36006	2.28188	0.43824	1.050	1.25386	1.60379	2.85765	0.34994
0.830	0.92863	1.36468	2.29332	0.43605	1.060	1.26996	1.61641	2.88637	0.34646
0.835	0.93547	1.36934	2.30482	0.43387	1.070	1.28619	1.62919	2.91538	0.34301
0.840	0.94233	1.37404	2.31637	0.43171	1.080	1.30254	1.64214	2.94468	0.33960
0.845	0.94921	1.37877	2.32798	0.42956	1.090	1.31903	1.65525	2.97428	0.33622
0.850	0.95612	1.38353	2.33965	0.42742	1.10	1.33565	1.66852	3.00417	0.33287
0.855	0.96305	1.38833	2.35138	0.42528	1.11	1.35240	1.68196	3.03436	0.32956
0.860	0.97000	1.39316	2.36316	0.42316	1.12	1.36929	1.69557	3.06485	0.32628
0.865	0.97698	1.39803	2.37501	0.42105	1.13	1.38631	1.70934	3.09566	0.32303
0.870	0.98398	1.40293	2.38691	0.41895	1.14	1.40347	1.72329	3.12677	0.31982
0.875	0.99101	1.40787	2.39888	0.41686	1.15	1.42078	1.73741	3.15819	0.31664
0.880	0.99806	1.41284	2.41090	0.41478	1.16	1.43822	1.75171	3.18993	0.31349
0.885	1.00514	1.41785	2.42298	0.41271	1.17	1.45581	1.76618	3.22199	0.31037
0.890	1.01224	1.42289	2.43513	0.41066	1.18	1.47355	1.78083	3.25437	0.30728
0.895	1.01936	1.42797	2.44734	0.40861	1.19	1.49143	1.79565	3.28708	0.30422
0.900	1.02652	1.43309	2.45960	0.40657	1.20	1.50946	1.81066	3.32012	0.30119
0.905	1.03370	1.43824	2.47193	0.40454	1.21	1.52764	1.82584	3.35349	0.29820
0.910	1.04090	1.44342	2.48432	0.40252	1.22	1.54598	1.84121	3.38719	0.29523
0.915	1.04813	1.44865	2.49678	0.40052	1.23	1.56447	1.85676	3.42123	0.29229
0.920	1.05539	1.45390	2.50929	0.39852	1.24	1.58311	1.87250	3.45561	0.28938
0.925	1.06267	1.45920	2.52187	0.39653	1.25	1.60192	1.88842	3.49034	0.28651
0.930	1.06998	1.46453	2.53451	0.39456	1.26	1.62088	1.90454	3.52542	0.28365
0.935	1.07731	1.46990	2.54721	0.39259	1.27	1.64001	1.92084	3.56085	0.28083
0.940	1.08468	1.47530	2.55998	0.39063	1.28	1.65930	1.93734	3.59664	0.27804
0.945	1.09207	1.48075	2.57281	0.38868	1.29	1.67876	1.95403	3.63279	0.27527
0.950	1.09948	1.48623	2.58571	0.38674	1.30	1.69838	1.97091	3.66930	0.27253
0.955	1.10693	1.49174	2.59867	0.38481	1.31	1.71818	1.98800	3.70617	0.26982
0.960	1.11440	1.49729	2.61170	0.38289	1.32	1.73814	2.00528	3.74342	0.26714
0.965	1.12190	1.50289	2.62479	0.38098	1.33	1.75828	2.02276	3.78104	0.26448
0.970	1.12943	1.50851	2.63795	0.37908	1.34	1.77860	2.04044	3.81905	0.26185
0.975	1.13699	1.51418	2.65117	0.37719	1.35	1.79909	2.05833	3.85743	0.25924
0.980	1.14457	1.51988	2.66446	0.37531	1.36	1.81977	2.07643	3.89619	0.25666
0.985	1.15219	1.52563	2.67781	0.37344	1.37	1.84062	2.09473	3.93535	0.25411
0.990	1.15983	1.53141	2.69123	0.37158	1.38	1.86166	2.11324	3.97490	0.25158
0.995	1.16750	1.53722	2.70472	0.36972	1.39	1.88289	2.13196	4.01485	0.24908

TABLE II — (Continued)
HYPERBOLIC AND EXPONENTIAL FUNCTIONS

x	$\sinh x$	$\cosh x$	e^x	e^{-x}	x	$\sinh x$	$\cosh x$	e^x	e^{-x}
1.40	1.90430	2.15090	4.05520	0.24660	1.80	2.94217	3.10747	6.04965	0.16530
1.41	1.92591	2.17005	4.09596	0.24414	1.81	2.97340	3.13705	6.11045	0.16365
1.42	1.94770	2.18942	4.13712	0.24171	1.82	3.00492	3.16694	6.17186	0.16203
1.43	1.96970	2.20900	4.17870	0.23931	1.83	3.03674	3.19715	6.23389	0.16041
1.44	1.99188	2.22881	4.22070	0.23693	1.84	3.06886	3.22768	6.29654	0.15882
1.45	2.01427	2.24884	4.26312	0.23457	1.85	3.10129	3.25853	6.35982	0.15724
1.46	2.03686	2.26910	4.30596	0.23224	1.86	3.13403	3.28970	6.42374	0.15567
1.47	2.05965	2.28958	4.34924	0.22993	1.87	3.16709	3.32121	6.48830	0.15412
1.48	2.08265	2.31029	4.39295	0.22764	1.88	3.20046	3.35305	6.55351	0.15259
1.49	2.10586	2.33123	4.43710	0.22537	1.89	3.23415	3.38522	6.61937	0.15107
1.50	2.12928	2.35241	4.48169	0.22313	1.90	3.26816	3.41773	6.68589	0.14957
1.51	2.15291	2.37382	4.52673	0.22091	1.91	3.30250	3.45058	6.75309	0.14808
1.52	2.17676	2.39547	4.57223	0.21872	1.92	3.33718	3.48378	6.82096	0.14661
1.53	2.20082	2.41736	4.61818	0.21654	1.93	3.37218	3.51733	6.88951	0.14515
1.54	2.22510	2.43949	4.66459	0.21438	1.94	3.40752	3.55123	6.95875	0.14370
1.55	2.24961	2.46186	4.71147	0.21225	1.95	3.44321	3.58548	7.02869	0.14227
1.56	2.27434	2.48448	4.75882	0.21014	1.96	3.47923	3.62009	7.09933	0.14086
1.57	2.29930	2.50735	4.80665	0.20806	1.97	3.51561	3.65507	7.17068	0.13946
1.58	2.32449	2.53047	4.85496	0.20598	1.98	3.55234	3.69041	7.24274	0.13807
1.59	2.34991	2.55384	4.90375	0.20393	1.99	3.58942	3.72611	7.31554	0.13670
1.60	2.37557	2.57746	4.95303	0.20190	2.00	3.62686	3.76220	7.38906	0.13534
1.61	2.40146	2.60135	5.00281	0.19989	2.01	3.66466	3.79865	7.46332	0.13399
1.62	2.42760	2.62549	5.05309	0.19790	2.02	3.70283	3.83549	7.53833	0.13266
1.63	2.45397	2.64990	5.10388	0.19593	2.03	3.74138	3.87271	7.61409	0.13134
1.64	2.48059	2.67457	5.15517	0.19398	2.04	3.78029	3.91032	7.69061	0.13003
1.65	2.50746	2.69951	5.20698	0.19205	2.05	3.81958	3.94832	7.76790	0.12874
1.66	2.53459	2.72472	5.25931	0.19014	2.06	3.85926	3.98671	7.84597	0.12745
1.67	2.56196	2.75021	5.31217	0.18825	2.07	3.89932	4.02550	7.92482	0.12619
1.68	2.58959	2.77596	5.36556	0.18637	2.08	3.93977	4.06470	8.00447	0.12493
1.69	2.61748	2.80200	5.41948	0.18452	2.09	3.98061	4.10430	8.08492	0.12369
1.70	2.64563	2.82832	5.47395	0.18268	2.10	4.02186	4.14431	8.16617	0.12246
1.71	2.67405	2.85491	5.52896	0.18087	2.11	4.06350	4.18474	8.24824	0.12124
1.72	2.70273	2.88180	5.58453	0.17907	2.12	4.10555	4.22558	8.33114	0.12003
1.73	2.73168	2.90897	5.64065	0.17728	2.13	4.14801	4.26685	8.41487	0.11884
1.74	2.76091	2.93643	5.69734	0.17552	2.14	4.19089	4.30855	8.49944	0.11766
1.75	2.79041	2.96419	5.75460	0.17377	2.15	4.23419	4.35067	8.58486	0.11649
1.76	2.82020	2.99224	5.81244	0.17205	2.16	4.27791	4.39323	8.67114	0.11533
1.77	2.85026	3.02059	5.87085	0.17033	2.17	4.32205	4.43623	8.75828	0.11418
1.78	2.88061	3.04925	5.92986	0.16864	2.18	4.36663	4.47967	8.84631	0.11304
1.79	2.91125	3.07821	5.98945	0.16696	2.19	4.41165	4.52356	8.93521	0.11192

TABLE II — (Continued)
HYPERBOLIC AND EXPONENTIAL FUNCTIONS

x	$\sinh x$	$\cosh x$	e^x	e^{-x}	x	$\sinh x$	$\cosh x$	e^x	e^{-x}
2.20	4.45711	4.56791	9.02501	0.11080	2.60	6.69473	6.76901	13.46374	0.07427
2.21	4.50301	4.61271	9.11572	0.10970	2.61	6.76276	6.83629	13.59905	0.07354
2.22	4.54936	4.65797	9.20733	0.10861	2.62	6.83146	6.90426	13.73573	0.07280
2.23	4.59617	4.70370	9.29987	0.10753	2.63	6.90085	6.97292	13.87377	0.07208
2.24	4.64344	4.74989	9.39331	0.10646	2.64	6.97092	7.04228	14.01320	0.07136
2.25	4.69117	4.79657	9.48774	0.10540	2.65	7.04169	7.11234	14.15404	0.07065
2.26	4.73937	4.84372	9.58309	0.10435	2.66	7.11317	7.18312	14.29629	0.06995
2.27	4.78804	4.89136	9.67940	0.10331	2.67	7.18536	7.25461	14.43997	0.06925
2.28	4.83720	4.93948	9.77668	0.10228	2.68	7.25827	7.32683	14.58509	0.06856
2.29	4.88684	4.98810	9.87494	0.10127	2.69	7.33190	7.39978	14.73168	0.06788
2.30	4.93696	5.03722	9.97418	0.10026	2.70	7.40626	7.47347	14.87973	0.06721
2.31	4.98758	5.08684	10.07443	0.09926	2.71	7.48137	7.54791	15.02928	0.06654
2.32	5.03870	5.13697	10.17567	0.09827	2.72	7.55722	7.62310	15.18032	0.06588
2.33	5.09032	5.18762	10.27794	0.09730	2.73	7.63383	7.69905	15.33289	0.06522
2.34	5.14245	5.23878	10.38124	0.09633	2.74	7.71121	7.77578	15.48699	0.06457
2.35	5.19510	5.29047	10.48557	0.09537	2.75	7.78935	7.85328	15.64263	0.06393
2.36	5.24827	5.34269	10.59095	0.09442	2.76	7.86828	7.93157	15.79984	0.06329
2.37	5.30196	5.39544	10.69739	0.09348	2.77	7.94799	8.01065	15.95864	0.06266
2.38	5.35618	5.44873	10.80490	0.09255	2.78	8.02849	8.09053	16.11902	0.06204
2.39	5.41093	5.50256	10.91349	0.09163	2.79	8.10980	8.17122	16.28102	0.06142
2.40	5.46623	5.55695	11.02318	0.09072	2.80	8.19192	8.25273	16.44465	0.06081
2.41	5.52207	5.61189	11.13396	0.08982	2.81	8.27486	8.33506	16.60992	0.06021
2.42	5.57847	5.66739	11.24586	0.08892	2.82	8.35862	8.41823	16.77685	0.05961
2.43	5.63542	5.72346	11.35888	0.08804	2.83	8.44322	8.50224	16.94546	0.05901
2.44	5.69294	5.78010	11.47304	0.08716	2.84	8.52867	8.58710	17.11577	0.05843
2.45	5.75103	5.83732	11.58835	0.08629	2.85	8.61497	8.67281	17.28778	0.05785
2.46	5.80969	5.89512	11.70481	0.08544	2.86	8.70213	8.75940	17.46153	0.05727
2.47	5.86893	5.95352	11.82245	0.08459	2.87	8.79016	8.84686	17.63702	0.05670
2.48	5.92876	6.01250	11.94126	0.08374	2.88	8.87907	8.93520	17.81427	0.05613
2.49	5.98918	6.07209	12.06128	0.08291	2.89	8.96887	9.02444	17.99331	0.05558
2.50	6.05020	6.13229	12.18250	0.08209	2.90	9.05956	9.11458	18.17415	0.05502
2.51	6.11183	6.19310	12.30493	0.08127	2.91	9.15116	9.20564	18.35680	0.05448
2.52	6.17407	6.25453	12.42860	0.08046	2.92	9.24368	9.29761	18.54129	0.05394
2.53	6.23692	6.31658	12.55351	0.07966	2.93	9.33712	9.39051	18.72763	0.05340
2.54	6.30040	6.37927	12.67967	0.07887	2.94	9.43149	9.48436	18.91585	0.05287
2.55	6.36451	6.44259	12.80710	0.07808	2.95	9.52681	9.57915	19.10595	0.05234
2.56	6.42926	6.50656	12.93582	0.07731	2.96	9.62308	9.67490	19.29797	0.05182
2.57	6.49464	6.57118	13.06583	0.07654	2.97	9.72031	9.77161	19.49192	0.05130
2.58	6.56068	6.63646	13.19714	0.07577	2.98	9.81851	9.86930	19.68782	0.05079
2.59	6.62738	6.70240	13.32977	0.07502	2.99	9.91770	9.96798	19.88568	0.05029

TABLE II — (Continued)
HYPERBOLIC AND EXPONENTIAL FUNCTIONS

z	$\sinh z$	$\cosh z$	e^z	e^{-z}	z	$\sinh z$	$\cosh z$	e^z	e^{-z}
3.00	10.0179	10.0677	20.08554	0.04979	4.50	45.0030	45.0141	90.01713	0.01111
3.05	10.5340	10.5814	21.11535	0.04736	4.55	47.3109	47.3215	94.63241	0.01057
3.10	11.0765	11.1215	22.19795	0.04505	4.60	49.7371	49.7472	99.48432	0.01005
3.15	11.6466	11.6895	23.33607	0.04285	4.65	52.2877	52.2973	104.58499	0.00956
3.20	12.2459	12.2866	24.53253	0.04076	4.70	54.9690	54.9781	109.94717	0.00910
3.25	12.8758	12.9146	25.79034	0.03877	4.75	57.7878	57.7965	115.58429	0.00865
3.30	13.5379	13.5748	27.11264	0.03688	4.80	60.7511	60.7593	121.51042	0.00823
3.35	14.2338	14.2689	28.50274	0.03509	4.85	63.8663	63.8741	127.74039	0.00783
3.40	14.9654	14.9987	29.96410	0.03337	4.90	67.1412	67.1486	134.28978	0.00745
3.45	15.7343	15.7661	31.50039	0.03175	4.95	70.5839	70.5910	141.17496	0.00708
3.50	16.5426	16.5728	33.11545	0.03020	5.00	74.2032	74.2099	148.41316	0.00674
3.55	17.3923	17.4210	34.81332	0.02873	5.05	78.0080	78.0144	156.02246	0.00641
3.60	18.2855	18.3128	36.59823	0.02732	5.10	82.0079	82.0140	164.02191	0.00610
3.65	19.2243	19.2503	38.47467	0.02599	5.15	86.2128	86.2186	172.43149	0.00580
3.70	20.2113	20.2360	40.44731	0.02472	5.20	90.6334	90.6389	181.27224	0.00552
3.75	21.2488	21.2723	42.52108	0.02352	5.25	95.281	95.286	190.56627	0.00525
3.80	22.3394	22.3618	44.70118	0.02237	5.30	100.166	100.171	200.33681	0.00499
3.85	23.4859	23.5072	46.99306	0.02128	5.35	105.320	105.307	210.60830	0.00475
3.90	24.6911	24.7113	49.40245	0.02024	5.40	110.701	110.706	221.40642	0.00452
3.95	25.9581	25.9773	51.93537	0.01926	5.45	116.377	116.381	232.75817	0.00430
4.00	27.2899	27.3082	54.59815	0.01832	5.50	122.344	122.348	244.69193	0.00409
4.05	28.6900	28.7074	57.39748	0.01742	5.55	128.617	128.621	257.23756	0.00389
4.10	30.1619	30.1784	60.34029	0.01657	5.60	135.211	135.215	270.42641	0.00370
4.15	31.7091	31.7249	63.43400	0.01576	5.65	142.144	142.148	284.29147	0.00352
4.20	33.3357	33.3507	66.68633	0.01499	5.70	149.432	149.435	298.86740	0.00335
4.25	35.0456	35.0593	70.10541	0.01426	5.75	157.094	157.097	314.19066	0.00318
4.30	36.8431	36.8567	73.69979	0.01357	5.80	165.148	165.151	330.29956	0.00303
4.35	38.7328	38.7457	77.47846	0.01291	5.85	173.616	173.619	347.23438	0.00288
4.40	40.7193	40.7316	81.45087	0.01228	5.90	182.517	182.520	365.03747	0.00274
4.45	42.8076	42.8193	85.62694	0.01168	5.95	191.875	191.878	383.75334	0.00261

TABLE III

RESISTANCE IN OHMS PER MILE, AT 25° C., OF STRANDED COPPER CONDUCTORS

Size of Conductor		Outside Diam. in Inches	Resistance Annealed Copper			Resistance Hard Drawn Copper		
Cir. Mils.	A.W.G. or B & S		D.C.	25 ~	60 ~	D.C.	25 ~	60 ~
2,000,000		1.631	0.02848	0.03032	0.03706	0.02925	0.03105	0.03771
1,900,000		1.590	0.02997	0.03174	0.03832	0.03079	0.03251	0.03899
1,800,000		1.548	0.03164	0.03332	0.03972	0.03250	0.03414	0.04045
1,700,000		1.504	0.03350	0.03509	0.04129	0.03441	0.03596	0.04207
1,600,000		1.459	0.03560	0.03710	0.04308	0.03656	0.03803	0.04392
1,500,000		1.412	0.03797	0.03938	0.04513	0.03900	0.04038	0.04602
1,400,000		1.364	0.04068	0.04201	0.04750	0.04179	0.04308	0.04847
1,300,000		1.315	0.04381	0.04504	0.05024	0.04500	0.04621	0.05131
1,200,000		1.263	0.04746	0.04860	0.05350	0.04875	0.04987	0.05466
1,100,000		1.209	0.05177	0.05282	0.05740	0.05318	0.05421	0.05868
1,000,000		1.152	0.05695	0.05791	0.06215	0.05850	0.05943	0.06358
950,000		1.123	0.05994	0.06085	0.06492	0.06158	0.06246	0.06643
900,000		1.093	0.06328	0.06414	0.06802	0.06500	0.06584	0.06963
850,000		1.062	0.06700	0.06781	0.07150	0.06882	0.06961	0.07323
800,000		1.031	0.07118	0.07196	0.07546	0.07312	0.07388	0.07729
750,000		0.998	0.07593	0.07665	0.07995	0.07800	0.07870	0.08192
700,000		0.964	0.08135	0.08203	0.08513	0.08357	0.08423	0.08725
650,000		0.929	0.08761	0.08824	0.09113	0.08999	0.09061	0.09343
600,000		0.891	0.09491	0.09549	0.09817	0.09749	0.09806	0.1007
550,000		0.853	0.1035	0.1041	0.1065	0.1064	0.1069	0.1093
500,000		0.814	0.1139	0.1144	0.1166	0.1170	0.1175	0.1197
450,000		0.772	0.1265	0.1270	0.1290	0.1300	0.1304	0.1324
400,000		0.725	0.1424	0.1427	0.1446	0.1462	0.1466	0.1484
350,000		0.679	0.1627	0.1630	0.1646	0.1671	0.1675	0.1690
300,000		0.628	0.1898	0.1901	0.1915	0.1950	0.1953	0.1966
250,000		0.574	0.2278	0.2280	0.2292	0.2340	0.2342	0.2354
211,600	0000	0.522	0.2691	0.2693	0.2703	0.2765	0.2767	0.2776
167,806	000	0.464	0.3394	0.3395	0.3403	0.3486	0.3488	0.3495
133,077	00	0.414	0.4279	0.4281	0.4287	0.4396	0.4398	0.4403
105,535	0	0.368	0.5396	0.5397	0.5402	0.5544	0.5545	0.5549
83,693	1	0.328	0.6805	0.6805	0.6809	0.6991	0.6991	0.6995
66,371	2	0.292	0.8581	0.8582	0.8584	0.8815	0.8816	0.8819
52,635	3	0.260	1.082	1.082	1.082	1.112	1.112	1.112
41,741	4	0.232	1.364	1.364	1.365	1.402	1.402	1.402
33,102	5	0.206	1.721	1.721	1.721	1.768	1.768	1.768

TABLE IV
RESISTANCE IN OHMS PER MILE, AT 25° C., OF ALUMINUM CABLE STEEL
REINFORCED (A.C.S.R.)

Size of Conductor	Out-side Diam.	No. of Wires		Resistance at a Current Density of 0 Amps. per Sq. In.			Resistance at 600 Amps. per Sq. In.		Resistance at 1200 Amps. per Sq. In.	
		Al.	St'l	D.C.	25 ~	60 ~	25 ~	60 ~	25 ~	60 ~
1,590,000	1.544	54	7	0.0587	0.0588	0.0591	0.0592	0.0607	0.0606	0.0662
1,510,500	1.506	54	7	0.0618	0.0619	0.0622	0.0623	0.0637	0.0636	0.0691
1,431,000	1.465	54	7	0.0652	0.0653	0.0656	0.0657	0.0671	0.0670	0.0724
1,351,500	1.424	54	7	0.0691	0.0692	0.0695	0.0695	0.0709	0.0708	0.0761
1,272,000	1.382	54	7	0.0734	0.0735	0.0738	0.0738	0.0752	0.0751	0.0802
1,192,500	1.337	54	7	0.0783	0.0784	0.0788	0.0787	0.0801	0.0799	0.0849
1,113,000	1.292	54	7	0.0839	0.0840	0.0844	0.0843	0.0857	0.0855	0.0903
1,033,500	1.246	54	7	0.0903	0.0905	0.0909	0.0908	0.0922	0.0919	0.0965
954,000	1.196	54	7	0.0979	0.0980	0.0982	0.0983	0.0997	0.0994	0.104
900,000	1.162	54	7	0.104	0.104	0.104	0.104	0.106	0.105	0.110
874,500	1.146	54	7	0.107	0.107	0.108	0.107	0.109	0.108	0.113
795,000	1.093	54	7	0.117	0.118	0.119	0.118	0.119	0.119	0.123
715,500	1.036	54	7	0.131	0.131	0.132	0.131	0.133	0.132	0.136
666,600	1.000	54	7	0.140	0.140	0.141	0.141	0.142	0.142	0.146
636,000	0.977	54	7	0.147	0.147	0.148	0.147	0.149	0.149	0.154
605,000	0.953	54	7	0.154	0.155	0.155	0.155	0.157	0.156	0.162
556,500	0.953	30	7	0.168	0.168	0.168	0.168	0.168	0.169	0.169
556,500	0.912	26	7	0.168	0.168	0.168	0.168	0.168	0.169	0.169
518,000	1.000	42	19	0.180	0.180	0.180	0.180	0.180	0.181	0.181
500,000	0.904	30	7	0.187	0.187	0.187	0.187	0.187	0.187	0.187
477,000	0.883	30	7	0.196	0.196	0.196	0.196	0.196	0.196	0.196
477,000	0.845	26	7	0.196	0.196	0.196	0.196	0.196	0.196	0.196
397,500	0.806	30	7	0.235	0.235	0.235	0.235	0.235	0.235	0.235
397,500	0.771	26	7	0.235	0.235	0.235	0.235	0.235	0.235	0.235
336,400	0.741	30	7	0.278	0.278	0.278	0.278	0.278	0.278	0.278
336,400	0.710	26	7	0.278	0.278	0.278	0.278	0.278	0.278	0.278
300,000	0.700	30	7	0.311	0.311	0.311	0.311	0.311	0.311	0.311
300,000	0.670	26	7	0.311	0.311	0.311	0.311	0.311	0.311	0.311
266,800	0.632	26	7	0.350	0.350	0.350	0.350	0.350	0.350	0.350
266,800	0.633	6	7	0.350	0.350	0.351	0.353	0.363	0.378	0.463

TABLE V
60-CYCLE INDUCTIVE REACTANCE IN OHMS PER MILE OF ONE STRANDED COPPER CONDUCTOR OF A SINGLE-PHASE OR OF AN EQUILATERAL THREE-PHASE LINE

Size of Conductor	Distance D between Centers of Conductors in Feet														
	3	3.5	4	5	6	7	8	9	11	13	15	17	19	21	23
Cir. Mils.															
2,000,000	0.400	0.509	0.525	0.552	0.574	0.593	0.609	0.624	0.648	0.668	0.686	0.701	0.714	0.727	0.738
1,900,000	0.404	0.512	0.528	0.555	0.578	0.596	0.613	0.627	0.651	0.671	0.689	0.704	0.717	0.730	0.741
1,800,000	0.407	0.515	0.532	0.559	0.581	0.600	0.616	0.630	0.654	0.675	0.692	0.707	0.721	0.733	0.744
1,700,000	0.410	0.519	0.535	0.562	0.584	0.603	0.619	0.634	0.658	0.678	0.696	0.711	0.724	0.736	0.747
1,600,000	0.414	0.523	0.539	0.566	0.588	0.607	0.623	0.637	0.662	0.682	0.699	0.714	0.728	0.740	0.751
1,500,000	0.418	0.527	0.543	0.570	0.592	0.611	0.627	0.641	0.666	0.686	0.703	0.719	0.732	0.744	0.755
1,400,000	0.422	0.531	0.547	0.574	0.596	0.615	0.631	0.646	0.670	0.690	0.708	0.723	0.736	0.748	0.759
1,300,000	0.426	0.535	0.552	0.579	0.601	0.620	0.636	0.650	0.674	0.695	0.712	0.727	0.741	0.753	0.764
1,200,000	0.430	0.540	0.557	0.584	0.606	0.624	0.641	0.655	0.679	0.700	0.717	0.732	0.746	0.758	0.769
1,100,000	0.434	0.544	0.562	0.589	0.611	0.630	0.646	0.660	0.685	0.705	0.722	0.737	0.751	0.763	0.774
1,000,000	0.438	0.548	0.566	0.593	0.615	0.634	0.650	0.664	0.689	0.710	0.728	0.744	0.757	0.769	0.780
950,000	0.442	0.552	0.570	0.597	0.619	0.638	0.654	0.668	0.693	0.714	0.732	0.747	0.760	0.772	0.783
900,000	0.446	0.556	0.574	0.601	0.624	0.643	0.659	0.673	0.697	0.717	0.735	0.750	0.763	0.775	0.787
850,000	0.450	0.560	0.578	0.605	0.628	0.647	0.663	0.677	0.701	0.721	0.738	0.753	0.767	0.779	0.790
800,000	0.454	0.564	0.582	0.609	0.631	0.650	0.666	0.680	0.704	0.725	0.742	0.757	0.771	0.783	0.794
750,000	0.458	0.568	0.586	0.613	0.635	0.654	0.670	0.684	0.708	0.729	0.746	0.761	0.775	0.787	0.798
700,000	0.462	0.572	0.590	0.617	0.639	0.658	0.674	0.688	0.712	0.733	0.750	0.765	0.779	0.791	0.802
650,000	0.466	0.576	0.594	0.621	0.643	0.662	0.678	0.693	0.717	0.737	0.755	0.770	0.783	0.795	0.806
600,000	0.470	0.580	0.598	0.625	0.647	0.666	0.682	0.696	0.720	0.740	0.758	0.773	0.786	0.798	0.809
550,000	0.474	0.584	0.602	0.629	0.651	0.670	0.686	0.700	0.724	0.744	0.762	0.777	0.790	0.801	0.812
500,000	0.478	0.588	0.606	0.633	0.655	0.674	0.690	0.704	0.728	0.748	0.766	0.781	0.794	0.806	0.817
450,000	0.482	0.592	0.610	0.637	0.659	0.678	0.694	0.708	0.732	0.752	0.770	0.785	0.800	0.812	0.823
400,000	0.486	0.596	0.614	0.641	0.663	0.682	0.698	0.712	0.736	0.756	0.774	0.789	0.804	0.816	0.827
350,000	0.490	0.600	0.618	0.645	0.667	0.686	0.702	0.716	0.740	0.760	0.778	0.793	0.808	0.820	0.831
300,000	0.494	0.604	0.622	0.649	0.671	0.690	0.706	0.720	0.744	0.764	0.782	0.797	0.812	0.824	0.835
250,000	0.498	0.608	0.626	0.653	0.675	0.694	0.710	0.724	0.748	0.768	0.786	0.801	0.816	0.828	0.839
211,800	0.502	0.612	0.630	0.657	0.679	0.698	0.714	0.728	0.752	0.772	0.790	0.805	0.820	0.832	0.843
167,800	0.506	0.616	0.634	0.661	0.683	0.702	0.718	0.732	0.756	0.776	0.794	0.809	0.824	0.836	0.847
133,077	0.510	0.620	0.638	0.665	0.687	0.706	0.722	0.736	0.760	0.780	0.798	0.813	0.828	0.840	0.851
105,535	0.514	0.624	0.642	0.669	0.691	0.710	0.726	0.740	0.764	0.784	0.802	0.817	0.832	0.844	0.855
83,693	0.518	0.628	0.646	0.673	0.695	0.714	0.730	0.744	0.768	0.788	0.806	0.821	0.836	0.848	0.859
66,371	0.522	0.632	0.650	0.677	0.699	0.718	0.734	0.748	0.772	0.792	0.810	0.825	0.840	0.852	0.863
52,635	0.526	0.636	0.654	0.681	0.703	0.722	0.738	0.752	0.776	0.796	0.814	0.829	0.844	0.856	0.867
41,741	0.530	0.640	0.658	0.685	0.707	0.726	0.742	0.756	0.780	0.800	0.818	0.833	0.848	0.860	0.871
33,102	0.534	0.644	0.662	0.689	0.711	0.730	0.746	0.760	0.784	0.804	0.822	0.837	0.852	0.864	0.875

NOTE. — For any three-phase arrangement of conductors, the distance D for which the values are tabulated is $D = (ABC)^{1/3}$ where A is the distance between lines 1 and 2; B is the distance between lines 2 and 3; C is the distance between lines 3 and 1.

TABLE VI
60-CYCLE INDUCTIVE REACTANCE IN OHMS PER MILE OF ONE STRANDED A.C.S.R. CONDUCTOR OF A
SINGLE-PHASE OR EQUILATERAL THREE-PHASE LINE

Size of Conductor	Number of Wires		Distance <i>D</i> between Centers of Conductors in Feet																
	Alum.	Steel	3.5	4	5	6	7	8	9	11	13	15	17	19	21	23	25	30	35
1,590,000	54	7	0.494	0.510	0.537	0.559	0.578	0.594	0.608	0.632	0.653	0.670	0.685	0.699	0.711	0.722	0.732	0.754	0.773
1,510,500	54	7	0.496	0.512	0.539	0.561	0.580	0.596	0.610	0.635	0.655	0.673	0.688	0.701	0.713	0.724	0.734	0.757	0.775
1,431,000	54	7	0.499	0.515	0.542	0.564	0.583	0.599	0.613	0.638	0.658	0.675	0.691	0.704	0.716	0.727	0.737	0.760	0.778
1,351,500	54	7	0.502	0.518	0.545	0.567	0.586	0.602	0.616	0.641	0.661	0.679	0.694	0.707	0.719	0.730	0.740	0.763	0.782
1,272,000	54	7	0.506	0.522	0.549	0.571	0.590	0.606	0.620	0.644	0.665	0.682	0.698	0.711	0.723	0.734	0.744	0.766	0.785
1,192,500	54	7	0.509	0.526	0.553	0.575	0.593	0.610	0.624	0.648	0.668	0.686	0.701	0.714	0.727	0.738	0.748	0.770	0.789
1,113,000	54	7	0.513	0.529	0.556	0.578	0.597	0.613	0.627	0.652	0.672	0.690	0.705	0.718	0.731	0.741	0.751	0.774	0.793
1,033,500	54	7	0.517	0.533	0.560	0.583	0.601	0.618	0.632	0.656	0.676	0.694	0.709	0.722	0.734	0.745	0.756	0.778	0.797
954,000	54	7	0.522	0.538	0.565	0.587	0.606	0.622	0.637	0.661	0.681	0.699	0.714	0.727	0.739	0.751	0.760	0.783	0.802
900,000	54	7	0.525	0.541	0.569	0.590	0.609	0.625	0.640	0.664	0.684	0.702	0.717	0.731	0.743	0.754	0.764	0.786	0.805
874,500	54	7	0.527	0.543	0.570	0.592	0.611	0.627	0.641	0.666	0.686	0.703	0.719	0.732	0.744	0.755	0.765	0.788	0.806
795,000	54	7	0.532	0.549	0.576	0.598	0.616	0.633	0.647	0.671	0.692	0.709	0.724	0.738	0.750	0.761	0.771	0.793	0.812
715,500	54	7	0.539	0.555	0.582	0.604	0.623	0.639	0.653	0.678	0.698	0.716	0.731	0.744	0.756	0.767	0.777	0.800	0.818
666,000	54	7	0.543	0.560	0.587	0.609	0.627	0.644	0.658	0.682	0.703	0.720	0.735	0.749	0.761	0.772	0.782	0.804	0.823
636,000	54	7	0.546	0.562	0.589	0.612	0.630	0.646	0.661	0.685	0.705	0.723	0.738	0.751	0.763	0.774	0.785	0.807	0.825
605,000	54	7	0.549	0.565	0.592	0.615	0.633	0.649	0.664	0.688	0.708	0.726	0.741	0.754	0.766	0.777	0.788	0.810	0.828
556,500	30	7	0.549	0.565	0.592	0.615	0.633	0.649	0.664	0.688	0.708	0.726	0.741	0.754	0.766	0.777	0.788	0.810	0.828
556,500	26	7	0.555	0.571	0.598	0.620	0.639	0.655	0.669	0.694	0.714	0.731	0.747	0.760	0.772	0.783	0.793	0.816	0.834
518,000	42	19	0.543	0.560	0.587	0.609	0.627	0.644	0.658	0.682	0.703	0.720	0.735	0.749	0.761	0.772	0.782	0.804	0.823
500,000	30	7	0.555	0.572	0.599	0.621	0.639	0.656	0.670	0.694	0.715	0.732	0.747	0.761	0.773	0.784	0.794	0.816	0.835
477,000	30	7	0.558	0.575	0.602	0.624	0.642	0.659	0.673	0.698	0.718	0.735	0.750	0.764	0.776	0.787	0.797	0.819	0.838
477,000	26	7	0.564	0.580	0.607	0.630	0.648	0.664	0.679	0.703	0.724	0.741	0.756	0.770	0.782	0.793	0.803	0.825	0.843
397,500	30	7	0.570	0.587	0.614	0.636	0.653	0.671	0.685	0.710	0.730	0.747	0.762	0.776	0.788	0.799	0.809	0.831	0.850
397,500	26	7	0.576	0.592	0.619	0.642	0.660	0.676	0.691	0.715	0.736	0.753	0.768	0.782	0.794	0.805	0.815	0.837	0.855
356,400	30	7	0.581	0.597	0.624	0.647	0.665	0.681	0.696	0.720	0.740	0.758	0.773	0.786	0.799	0.809	0.820	0.842	0.860
336,400	26	7	0.587	0.603	0.630	0.652	0.671	0.687	0.701	0.726	0.746	0.763	0.779	0.792	0.804	0.815	0.825	0.848	0.866
300,000	30	7	0.589	0.605	0.632	0.654	0.673	0.689	0.703	0.728	0.748	0.765	0.780	0.794	0.806	0.817	0.827	0.849	0.868
300,000	26	7	0.595	0.611	0.638	0.660	0.679	0.695	0.709	0.734	0.754	0.771	0.786	0.800	0.812	0.823	0.833	0.855	0.874
266,800	26	7	0.600	0.616	0.644	0.666	0.684	0.701	0.715	0.739	0.760	0.777	0.792	0.806	0.818	0.829	0.839	0.861	0.880

NOTE.— For any three-phase arrangement of conductors, the distance D for which the values are tabulated is $D = (ABC)^{1/3}$ where A is the distance between lines 1 and 2; B is the distance between lines 2 and 3; C is the distance between lines 3 and 1.

TABLE VII
60-CYCLE CAPACITIVE SUSCEPTANCE TO NEUTRAL IN MICROMHOS PER MILE OF ONE STRANDED BARE COPPER
CONDUCTOR OF A SINGLE-PHASE OR EQUILATERAL THREE-PHASE LINE

Size of Conductor	Distance <i>D</i> between Centers of Conductors in Feet													
	3	3.5	4	5	6	7	8	9	11	13	15	17	19	21
Cir. Mils.														
2,000,000	8.90	8.55	8.27	7.84	7.52	7.27	7.07	6.90	6.63	6.42	6.25	6.10	5.98	5.88
1,900,000	8.84	8.50	8.22	7.80	7.48	7.23	7.03	6.86	6.59	6.38	6.22	6.05	5.93	5.85
1,800,000	8.78	8.44	8.17	7.75	7.44	7.19	6.99	6.83	6.56	6.35	6.19	6.05	5.93	5.85
1,700,000	8.71	8.38	8.11	7.70	7.39	7.15	6.95	6.79	6.52	6.32	6.15	6.02	5.90	5.80
1,600,000	8.65	8.32	8.05	7.64	7.34	7.10	6.91	6.74	6.48	6.28	6.12	5.98	5.87	5.77
1,500,000	8.57	8.25	7.99	7.59	7.29	7.05	6.86	6.70	6.44	6.24	6.08	5.95	5.83	5.73
1,400,000	8.50	8.18	7.92	7.53	7.23	7.00	6.81	6.66	6.40	6.20	6.05	5.91	5.80	5.61
1,300,000	8.42	8.11	7.86	7.47	7.18	6.95	6.76	6.61	6.36	6.16	6.01	5.87	5.76	5.58
1,200,000	8.34	8.03	7.78	7.40	7.12	6.89	6.71	6.56	6.31	6.12	5.96	5.83	5.72	5.63
1,100,000	8.25	7.95	7.71	7.33	7.05	6.83	6.65	6.50	6.26	6.07	5.92	5.79	5.68	5.59
1,000,000	8.15	7.86	7.62	7.26	6.98	6.77	6.59	6.44	6.20	6.02	5.87	5.74	5.64	5.54
950,000	8.10	7.81	7.58	7.22	6.94	6.73	6.56	6.41	6.17	5.99	5.84	5.72	5.61	5.52
900,000	8.05	7.76	7.53	7.17	6.91	6.69	6.52	6.38	6.14	5.96	5.81	5.68	5.59	5.50
850,000	7.99	7.71	7.48	7.13	6.87	6.66	6.49	6.34	6.11	5.93	5.79	5.66	5.56	5.47
800,000	7.94	7.66	7.43	7.09	6.82	6.62	6.45	6.31	6.08	5.90	5.76	5.64	5.53	5.44
750,000	7.88	7.60	7.38	7.04	6.78	6.58	6.41	6.27	6.04	5.87	5.72	5.61	5.50	5.41
700,000	7.81	7.53	7.33	6.99	6.73	6.53	6.37	6.23	6.01	5.83	5.69	5.57	5.47	5.38
650,000	7.75	7.48	7.27	6.93	6.68	6.49	6.32	6.19	5.97	5.79	5.66	5.54	5.44	5.35
600,000	7.68	7.41	7.20	6.88	6.63	6.43	6.27	6.14	5.92	5.75	5.62	5.50	5.40	5.32
550,000	7.60	7.34	7.14	6.81	6.57	6.38	6.22	6.09	5.88	5.71	5.58	5.46	5.37	5.28
500,000	7.52	7.27	7.07	6.75	6.51	6.32	6.17	6.04	5.83	5.67	5.53	5.42	5.33	5.24
450,000	7.43	7.19	6.99	6.68	6.45	6.26	6.11	5.98	5.78	5.62	5.49	5.38	5.28	5.13
400,000	7.33	7.09	6.90	6.67	6.37	6.19	6.04	5.92	5.72	5.56	5.43	5.32	5.23	5.15
350,000	7.23	7.00	6.81	6.51	6.29	6.12	5.97	5.85	5.65	5.50	5.37	5.27	5.18	5.10
300,000	7.11	6.89	6.70	6.42	6.20	6.03	5.89	5.77	5.58	5.43	5.31	5.20	5.12	5.04
250,000	6.97	6.76	6.58	6.31	6.10	5.93	5.80	5.68	5.50	5.35	5.23	5.13	5.05	4.97
211,600	6.84	6.63	6.46	6.20	6.00	5.84	5.70	5.59	5.41	5.27	5.16	5.07	4.98	4.90
167,806	6.68	6.49	6.32	6.07	5.88	5.72	5.59	5.49	5.31	5.18	5.07	4.97	4.89	4.82
133,077	6.53	6.34	6.19	5.94	5.76	5.61	5.49	5.39	5.22	5.09	4.98	4.89	4.81	4.73
105,535	6.39	6.21	6.06	5.83	5.65	5.51	5.39	5.29	5.13	5.00	4.90	4.81	4.73	4.67
83,693	6.25	6.08	5.94	5.71	5.54	5.40	5.29	5.19	5.04	4.92	4.81	4.73	4.66	4.59
66,371	6.12	5.95	5.82	5.60	5.44	5.30	5.19	5.10	4.95	4.83	4.74	4.65	4.58	4.52
52,635	5.99	5.83	5.70	5.50	5.34	5.21	5.10	5.01	4.87	4.75	4.66	4.58	4.51	4.45
41,741	5.87	5.72	5.59	5.39	5.24	5.12	5.02	4.93	4.79	4.68	4.59	4.51	4.44	4.39
33,102	5.76	5.61	5.49	5.29	5.15	5.03	4.93	4.85	4.71	4.60	4.52	4.44	4.38	4.32

NOTE. — For any three-phase arrangement of conductors, the distance *D* for which the values are tabulated is $D = (ABC)^{1/3}$ where *A* is the distance between lines 1 and 2; *B* is the distance between lines 2 and 3; *C* is the distance between lines 3 and 1.

TABLE VIII
60-CYCLE CAPACITIVE SUSCEPTANCE TO NEUTRAL IN MICROMHOS PER MILE OF ONE STRANDED BARE
A.C.S.R. CONDUCTOR OF A SINGLE-PHASE OR EQUILATERAL THREE-PHASE LINE

Size of Conductor	Distance D between Centers of Conductors in Feet														
	Number of Wires														
	Alum.	Steel	3.5	4	5	6	7	8	9	11	13	15	17	19	21
Cir. Mils.															
1,500,000	54	7	8.43	8.16	7.74	7.43	7.19	6.99	6.82	6.56	6.35	6.18	6.04	5.93	5.83
1,510,500	54	7	8.38	8.11	7.70	7.39	7.15	6.95	6.79	6.52	6.32	6.15	6.02	5.90	5.80
1,431,000	54	7	8.32	8.06	7.65	7.35	7.11	6.91	6.75	6.49	6.29	6.12	5.99	5.87	5.77
1,351,500	54	7	8.27	8.00	7.60	7.30	7.07	6.87	6.71	6.45	6.25	6.09	5.96	5.84	5.74
1,272,000	54	7	8.21	7.95	7.55	7.25	7.02	6.83	6.67	6.41	6.22	6.06	5.93	5.81	5.72
1,192,500	54	7	8.14	7.89	7.50	7.20	6.98	6.79	6.63	6.38	6.18	6.02	5.89	5.78	5.68
1,113,000	54	7	8.08	7.82	7.44	7.15	6.93	6.74	6.59	6.34	6.15	5.99	5.86	5.75	5.65
1,033,500	54	7	8.00	7.76	7.38	7.10	6.87	6.69	6.54	6.30	6.10	5.95	5.82	5.71	5.61
954,000	54	7	7.93	7.69	7.31	7.04	6.82	6.64	6.49	6.25	6.06	5.90	5.78	5.67	5.58
900,000	54	7	7.88	7.63	7.27	6.99	6.78	6.60	6.45	6.21	6.03	5.88	5.75	5.64	5.55
874,500	54	7	7.85	7.61	7.25	6.98	6.76	6.58	6.43	6.19	6.01	5.86	5.74	5.63	5.54
795,000	54	7	7.76	7.53	7.17	6.90	6.69	6.52	6.38	6.14	5.96	5.81	5.69	5.59	5.49
715,500	54	7	7.67	7.44	7.09	6.83	6.62	6.45	6.31	6.09	5.90	5.76	5.64	5.54	5.45
666,500	54	7	7.61	7.39	7.04	6.78	6.58	6.41	6.27	6.04	5.87	5.73	5.61	5.50	5.42
636,000	54	7	7.57	7.35	7.01	6.75	6.55	6.38	6.24	6.02	5.84	5.70	5.59	5.49	5.39
605,000	54	7	7.53	7.31	6.97	6.72	6.52	6.35	6.22	5.99	5.82	5.68	5.56	5.46	5.38
556,500	30	7	7.53	7.31	6.97	6.72	6.52	6.35	6.22	5.99	5.82	5.68	5.56	5.46	5.38
556,500	26	7	7.45	7.24	6.91	6.66	6.46	6.30	6.16	5.95	5.78	5.64	5.52	5.43	5.34
518,000	42	19	7.61	7.39	7.04	6.78	6.58	6.41	6.27	6.04	5.87	5.73	5.61	5.50	5.42
500,000	30	7	7.44	7.22	6.90	6.65	6.45	6.29	6.16	5.94	5.77	5.63	5.52	5.42	5.34
477,000	30	7	7.40	7.19	6.86	6.62	6.42	6.26	6.13	5.91	5.75	5.61	5.49	5.40	5.31
477,000	26	7	7.33	7.12	6.80	6.56	6.37	6.21	6.08	5.87	5.70	5.57	5.46	5.36	5.27
397,500	30	7	7.25	7.05	6.74	6.50	6.31	6.16	6.03	5.82	5.66	5.52	5.41	5.32	5.24
397,500	26	7	7.19	6.99	6.68	6.45	6.26	6.11	5.98	5.78	5.61	5.49	5.38	5.28	5.20
336,400	30	7	7.13	6.93	6.63	6.40	6.22	6.07	5.94	5.74	5.58	5.45	5.34	5.25	5.17
336,400	26	7	7.06	6.87	6.57	6.35	6.16	6.02	5.89	5.69	5.54	5.41	5.30	5.21	5.14
300,000	30	7	7.04	6.85	6.55	6.33	6.15	6.01	5.88	5.68	5.53	5.40	5.29	5.20	5.12
300,000	26	7	6.98	6.79	6.50	6.20	6.10	5.96	5.84	5.64	5.49	5.36	5.26	5.17	5.09
266,800	26	7	6.89	6.71	6.42	6.21	6.04	5.90	5.78	5.58	5.44	5.31	5.21	5.12	5.04
266,800	6	7	6.90	6.71	6.43	6.21	6.04	5.90	5.78	5.59	5.44	5.31	5.21	5.12	5.04

NOTE. — For any three-phase arrangement of conductors, the distance D for which the values are tabulated is $D = (ABC)^{1/2}$ where A is the distance between lines 1 and 2; B is the distance between lines 2 and 3; C is the distance between lines 3 and 1.

TABLE IX_a
FUNDAMENTAL ($n=1$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
0	0	0	0.0000		1.0000				
1	5	5	0.0872		0.9962				
2	10	10	0.1737		0.9848				
3	15	15	0.2588		0.9659				
4	20	20	0.3420		0.9397				
5	25	25	0.4226		0.9063				
6	30	30	0.5000		0.8660				
7	35	35	0.5736		0.8192				
8	40	40	0.6428		0.7660				
9	45	45	0.7071		0.7071				
10	50	50	0.7660		0.6428				
11	55	55	0.8192		0.5736				
12	60	60	0.8660		0.5000				
13	65	65	0.9063		0.4226				
14	70	70	0.9397		0.3420				
15	75	75	0.9659		0.2588				
16	80	80	0.9848		0.1737				
17	85	85	0.9962		0.0872				
18	90	90	1.0000		0.0000				
19	95	95	0.9962		-0.0872				
20	100	100	0.9848		-0.1737				
21	105	105	0.9659		-0.2588				
22	110	110	0.9397		-0.3420				
23	115	115	0.9063		-0.4226				
24	120	120	0.8660		-0.5000				
25	125	125	0.8192		-0.5736				
26	130	130	0.7660		-0.6428				
27	135	135	0.7071		-0.7071				
28	140	140	0.6428		-0.7660				
29	145	145	0.5736		-0.8192				
30	150	150	0.5000		-0.8660				
31	155	155	0.4226		-0.9063				
32	160	160	0.3420		-0.9397				
33	165	165	0.2588		-0.9659				
34	170	170	0.1737		-0.9848				
35	175	175	0.0872		-0.9962				
36	180	180	0.0000		-1.0000				

TABLE IXb
FUNDAMENTAL ($n=1$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
37	185	185	-0.0872		-0.9962				
38	190	190	-0.1737		-0.9848				
39	195	195	-0.2588		-0.9659				
40	200	200	-0.3420		-0.9397				
41	205	205	-0.4226		-0.9063				
42	210	210	-0.5000		-0.8660				
43	215	215	-0.5736		-0.8192				
44	220	220	-0.6428		-0.7660				
45	225	225	-0.7071		-0.7071				
46	230	230	-0.7660		-0.6428				
47	235	235	-0.8192		-0.5736				
48	240	240	-0.8660		-0.5000				
49	245	245	-0.9063		-0.4226				
50	250	250	-0.9397		-0.3420				
51	255	255	-0.9659		-0.2588				
52	260	260	-0.9848		-0.1737				
53	265	265	-0.9962		-0.0872				
54	270	270	-1.0000		-0.0000				
55	275	275	-0.9962		+0.0872				
56	280	280	-0.9848		0.1737				
57	285	285	-0.9659		0.2588				
58	290	290	-0.9397		0.3420				
59	295	295	-0.9063		0.4226				
60	300	300	-0.8660		0.5000				
61	305	305	-0.8192		0.5736				
62	310	310	-0.7660		0.6428				
63	315	315	-0.7071		0.7071				
64	320	320	-0.6428		0.7660				
65	325	325	-0.5736		0.8192				
66	330	330	-0.5000		0.8660				
67	335	335	-0.4226		0.9063				
68	340	340	-0.3420		0.9397				
69	345	345	-0.2588		0.9659				
70	350	350	-0.1737		0.9848				
71	355	355	-0.0872		0.9962				
72	360	360	-0.0000		1.0000				

TABLE X_a
SECOND HARMONIC ($n=2$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
0	0	0	0.0000		1.0000				
1	5	10	0.1737		0.9848				
2	10	20	0.3420		0.9397				
3	15	30	0.5000		0.8660				
4	20	40	0.6428		0.7660				
5	25	50	0.7660		0.6428				
6	30	60	0.8660		0.5000				
7	35	70	0.9397		0.3420				
8	40	80	0.9848		0.1737				
9	45	90	1.0000		0.0000				
10	50	100	0.9848		-0.1737				
11	55	110	0.9397		-0.3420				
12	60	120	0.8660		-0.5000				
13	65	130	0.7660		-0.6428				
14	70	140	0.6428		-0.7660				
15	75	150	0.5000		-0.8660				
16	80	160	0.3420		-0.9397				
17	85	170	0.1737		-0.9848				
18	90	180	0.0000		-1.0000				
19	95	190	-0.1737		-0.9848				
20	100	200	-0.3420		-0.9397				
21	105	210	-0.5000		-0.8660				
22	110	220	-0.6428		-0.7660				
23	115	230	-0.7660		-0.6428				
24	120	240	-0.8660		-0.5000				
25	125	250	-0.9397		-0.3420				
26	130	260	-0.9848		-0.1737				
27	135	270	-1.0000		-0.0000				
28	140	280	-0.9848		+0.1737				
29	145	290	-0.9397		0.3420				
30	150	300	-0.8660		0.5000				
31	155	310	-0.7660		0.6428				
32	160	320	-0.6428		0.7660				
33	165	330	-0.5000		0.8660				
34	170	340	-0.3420		0.9397				
35	175	350	-0.1737		0.9848				
36	180	360	-0.0000		1.0000				

TABLE Xb
SECOND HARMONIC ($n=2$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
37	185	10	0.1737		0.9848				
38	190	20	0.3420		0.9397				
39	195	30	0.5000		0.8660				
40	200	40	0.6428		0.7660				
41	205	50	0.7660		0.6428				
42	210	60	0.8660		0.5000				
43	215	70	0.9397		0.3420				
44	220	80	0.9848		0.1737				
45	225	90	1.0000		0.0000				
46	230	100	0.9848		-0.1737				
47	235	110	0.9397		-0.3420				
48	240	120	0.8660		-0.5000				
49	245	130	0.7660		-0.6428				
50	250	140	0.6428		-0.7660				
51	255	150	0.5000		-0.8660				
52	260	160	0.3420		-0.9397				
53	265	170	0.1737		-0.9848				
54	270	180	0.0000		-1.0000				
55	275	190	-0.1737		-0.9848				
56	280	200	-0.3420		-0.9397				
57	285	210	-0.5000		-0.8660				
58	290	220	-0.6428		-0.7660				
59	295	230	-0.7660		-0.6428				
60	300	240	-0.8660		-0.5000				
61	305	250	-0.9397		-0.3420				
62	310	260	-0.9848		-0.1737				
63	315	270	-1.0000		-0.0000				
64	320	280	-0.9848		+0.1737				
65	325	290	-0.9397		0.3420				
66	330	300	-0.8660		0.5000				
67	335	310	-0.7660		0.6428				
68	340	320	-0.6428		0.7660				
69	345	330	-0.5000		0.8660				
70	350	340	-0.3420		0.9397				
71	355	350	-0.1737		0.9848				
72	360	360	-0.0000		1.0000				

TABLE XIa
THIRD HARMONIC ($n=3$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
0	0	0	0.0000		1.0000				
1	5	15	0.2588		0.9659				
2	10	30	0.5000		0.8660				
3	15	45	0.7071		0.7071				
4	20	60	0.8660		0.5000				
5	25	75	0.9659		0.2588				
6	30	90	1.0000		0.0000				
7	35	105	0.9659		-0.2588				
8	40	120	0.8660		-0.5000				
9	45	135	0.7071		-0.7071				
10	50	150	0.5000		-0.8660				
11	55	165	0.2588		-0.9659				
12	60	180	0.0000		-1.0000				
13	65	195	-0.2588		-0.9659				
14	70	210	-0.5000		-0.8660				
15	75	225	-0.7071		-0.7071				
16	80	240	-0.8660		-0.5000				
17	85	255	-0.9659		-0.2588				
18	90	270	-1.0000		0.0000				
19	95	285	-0.9659		+0.2588				
20	100	300	-0.8660		0.5000				
21	105	315	-0.7071		0.7071				
22	110	330	-0.5000		0.8660				
23	115	345	-0.2588		0.9659				
24	120	360	0.0000		1.0000				
25	125	15	0.2588		0.9659				
26	130	30	0.5000		0.8660				
27	135	45	0.7071		0.7071				
28	140	60	0.8660		0.5000				
29	145	75	0.9659		0.2588				
30	150	90	1.0000		0.0000				
31	155	105	0.9659		-0.2588				
32	160	120	0.8660		-0.5000				
33	165	135	0.7071		-0.7071				
34	170	150	0.5000		-0.8660				
35	175	165	0.2588		-0.9659				
36	180	180	0.0000		-1.0000				

TABLE XIb
THIRD HARMONIC ($n=3$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
37	185	195	-0.2588		-0.9659				
38	190	210	-0.5000		-0.8660				
39	195	225	-0.7071		-0.7071				
40	200	240	-0.8660		-0.5000				
41	205	255	-0.9659		-0.2588				
42	210	270	-1.0000		0.0000				
43	215	285	-0.9659		+0.2588				
44	220	300	-0.8660		0.5000				
45	225	315	-0.7071		0.7071				
46	230	330	-0.5000		0.8660				
47	235	345	-0.2588		0.9659				
48	240	360	0.0000		1.0000				
49	245	15	0.2588		0.9659				
50	250	30	0.5000		0.8660				
51	255	45	0.7071		0.7071				
52	260	60	0.8660		0.5000				
53	265	75	0.9659		0.2588				
54	270	90	1.0000		0.0000				
55	275	105	0.9659		-0.2588				
56	280	120	0.8660		-0.5000				
57	285	135	0.7071		-0.7071				
58	290	150	0.5000		-0.8660				
59	295	165	0.2588		-0.9659				
60	300	180	0.0000		-1.0000				
61	305	195	-0.2588		-0.9659				
62	310	210	-0.5000		-0.8660				
63	315	225	-0.7071		-0.7071				
64	320	240	-0.8660		-0.5000				
65	325	255	-0.9659		-0.2588				
66	330	270	-1.0000		0.0000				
67	335	285	-0.9659		+0.2588				
68	340	300	-0.8660		0.5000				
69	345	315	-0.7071		0.7071				
70	350	330	-0.5000		0.8660				
71	355	345	-0.2588		0.9659				
72	360	360	0.0000		1.0000				

TABLE XIIa
FOURTH HARMONIC ($n=4$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
0	0	0	0.0000		1.0000				
1	5	20	0.3420		0.9397				
2	10	40	0.6428		0.7660				
3	15	60	0.8660		0.5000				
4	20	80	0.9848		0.1737				
5	25	100	0.9848		-0.1737				
6	30	120	0.8660		-0.5000				
7	35	140	0.6428		-0.7660				
8	40	160	0.3420		-0.9397				
9	45	180	0.0000		-1.0000				
10	50	200	-0.3420		-0.9397				
11	55	220	-0.6428		-0.7660				
12	60	240	-0.8660		-0.5000				
13	65	260	-0.9848		-0.1737				
14	70	280	-0.9848		+0.1737				
15	75	300	-0.8660		0.5000				
16	80	320	-0.6428		0.7660				
17	85	340	-0.3420		0.9397				
18	90	360	0.0000		1.0000				
19	95	20	0.3420		0.9397				
20	100	40	0.6428		0.7660				
21	105	60	0.8660		0.5000				
22	110	80	0.9848		0.1737				
23	115	100	0.9848		-0.1737				
24	120	120	0.8660		-0.5000				
25	125	140	0.6428		-0.7660				
26	130	160	0.3420		-0.9397				
27	135	180	0.0000		-1.0000				
28	140	200	-0.3420		-0.9397				
29	145	220	-0.6428		-0.7660				
30	150	240	-0.8660		-0.5000				
31	155	260	-0.9848		-0.1737				
32	160	280	-0.9848		+0.1737				
33	165	300	-0.8660		0.5000				
34	170	320	-0.6428		0.7660				
35	175	340	-0.3420		0.9397				
36	180	360	0.0000		1.0000				

TABLE XIIb
FOURTH HARMONIC ($n=4$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
37	185	20	0.3420		0.9397				
38	190	40	0.6428		0.7660				
39	195	60	0.8660		0.5000				
40	200	80	0.9848		0.1737				
41	205	100	0.9848		-0.1737				
42	210	120	0.8660		-0.5000				
43	215	140	0.6428		-0.7660				
44	220	160	0.3420		-0.9397				
45	225	180	0.0000		-1.0000				
46	230	200	-0.3420		-0.9397				
47	235	220	-0.6428		-0.7660				
48	240	240	-0.8660		-0.5000				
49	245	260	-0.9848		-0.1737				
50	250	280	-0.9848		+0.1737				
51	255	300	-0.8660		0.5000				
52	260	320	-0.6428		0.7660				
53	265	340	-0.3420		0.9397				
54	270	360	0.0000		1.0000				
55	275	20	0.3420		0.9397				
56	280	40	0.6428		0.7660				
57	285	60	0.8660		0.5000				
58	290	80	0.9848		0.1737				
59	295	100	0.9848		-0.1737				
60	300	120	0.8660		-0.5000				
61	305	140	0.6428		-0.7660				
62	310	160	0.3420		-0.9397				
63	315	180	0.0000		-1.0000				
64	320	200	-0.3420		-0.9397				
65	325	220	-0.6428		-0.7660				
66	330	240	-0.8660		-0.5000				
67	335	260	-0.9848		-0.1737				
68	340	280	-0.9848		+0.1737				
69	345	300	-0.8660		0.5000				
70	350	320	-0.6428		0.7660				
71	355	340	-0.3420		0.9397				
72	360	360	0.0000		1.0000				

TABLE XIIIa
FIFTH HARMONIC ($n=5$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
0	0	0	0.0000		1.0000				
1	5	25	0.4226		0.9063				
2	10	50	0.7660		0.6428				
3	15	75	0.9659		0.2588				
4	20	100	0.9848		-0.1737				
5	25	125	0.8192		-0.5736				
6	30	150	0.5000		-0.8660				
7	35	175	0.0872		-0.9962				
8	40	200	-0.3420		-0.9397				
9	45	225	-0.7071		-0.7071				
10	50	250	-0.9397		-0.3420				
11	55	275	-0.9962		+0.0872				
12	60	300	-0.8660		0.5000				
13	65	325	-0.5736		0.8192				
14	70	350	-0.1737		0.9848				
15	75	15	+0.2588		0.9659				
16	80	40	0.6428		0.7660				
17	85	65	0.9063		0.4226				
18	90	90	1.0000		0.0000				
19	95	115	0.9063		-0.4226				
20	100	140	0.6428		-0.7660				
21	105	165	0.2588		-0.9659				
22	110	190	-0.1737		-0.9848				
23	115	215	-0.5736		-0.8192				
24	120	240	-0.8660		-0.5000				
25	125	265	-0.9962		-0.0872				
26	130	290	-0.9397		+0.3420				
27	135	315	-0.7071		0.7071				
28	140	340	-0.3420		0.9397				
29	145	5	+0.0872		0.9962				
30	150	30	0.5000		0.8660				
31	155	55	0.8192		0.5736				
32	160	80	0.9848		0.1737				
33	165	105	0.9659		-0.2588				
34	170	130	0.7660		-0.6428				
35	175	155	0.4226		-0.9063				
36	180	180	0.0000		-1.0000				

TABLE XIIIb
FIFTH HARMONIC ($n=5$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
37	185	205	-0.4226		-0.9063				
38	190	230	-0.7660		-0.6428				
39	195	255	-0.9659		-0.2588				
40	200	280	-0.9848		+0.1737				
41	205	305	-0.8192		0.5736				
42	210	330	-0.5000		0.8660				
43	215	355	-0.0872		0.9962				
44	220	20	+0.3420		0.9397				
45	225	45	0.7071		0.7071				
46	230	70	0.9397		0.3420				
47	235	95	0.9962		-0.0872				
48	240	120	0.8660		-0.5000				
49	245	145	0.5737		-0.8192				
50	250	170	0.1737		-0.9848				
51	255	195	-0.2588		-0.9659				
52	260	220	-0.6428		-0.7660				
53	265	245	-0.9063		-0.4226				
54	270	270	-1.0000		0.0000				
55	275	295	-0.9063		+0.4226				
56	280	320	-0.6428		0.7660				
57	285	345	-0.2588		0.9659				
58	290	10	+0.1737		0.9848				
59	295	35	0.5736		0.8192				
60	300	60	0.8660		0.5000				
61	305	85	0.9962		0.0872				
62	310	110	0.9397		-0.3420				
63	315	135	0.7071		-0.7071				
64	320	160	0.3420		-0.9397				
65	325	185	-0.0872		-0.9962				
66	330	210	-0.5000		-0.8660				
67	335	235	-0.8192		-0.5736				
68	340	260	-0.9848		-0.1737				
69	345	285	-0.9659		+0.2588				
70	350	310	-0.7660		0.6428				
71	355	335	-0.4226		0.9063				
72	360	360	0.0000		1.0000				

TABLE XIV_a
SIXTH HARMONIC ($n=6$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
0	0	0	0.0000		1.0000				
1	5	30	0.5000		0.8660				
2	10	60	0.8660		0.5000				
3	15	90	1.0000		0.0000				
4	20	120	0.8660		-0.5000				
5	25	150	0.5000		-0.8660				
6	30	180	0.0000		-1.0000				
7	35	210	-0.5000		-0.8660				
8	40	240	-0.8660		-0.5000				
9	45	270	-1.0000		0.0000				
10	50	300	-0.8660		+0.5000				
11	55	330	-0.5000		+0.8660				
12	60	360	0.0000		1.0000				
13	65	30	0.5000		0.8660				
14	70	60	0.8660		0.5000				
15	75	90	1.0000		0.0000				
16	80	120	0.8660		-0.5000				
17	85	150	0.5000		-0.8660				
18	90	180	0.0000		-1.0000				
19	95	210	-0.5000		-0.8660				
20	100	240	-0.8660		-0.5000				
21	105	270	-1.0000		0.0000				
22	110	300	-0.8660		+0.5000				
23	115	330	-0.5000		0.8660				
24	120	360	0.0000		1.0000				
25	125	30	+0.5000		0.8660				
26	130	60	0.8660		0.5000				
27	135	90	1.0000		0.0000				
28	140	120	0.8660		-0.5000				
29	145	150	0.5000		-0.8660				
30	150	180	0.0000		-1.0000				
31	155	210	-0.5000		-0.8660				
32	160	240	-0.8660		-0.5000				
33	165	270	-1.0000		0.0000				
34	170	300	-0.8660		+0.5000				
35	175	330	-0.5000		0.8660				
36	180	360	0.0000		1.0000				

TABLE XIVb
SIXTH HARMONIC ($n=6$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
37	185	30	0.5000		0.8660				
38	190	60	0.8660		0.5000				
39	195	90	1.0000		0.0000				
40	200	120	0.8660		-0.5000				
41	205	150	0.5000		-0.8660				
42	210	180	0.0000		-1.0000				
43	215	210	-0.5000		-0.8660				
44	220	240	-0.8660		-0.5000				
45	225	270	-1.0000		0.0000				
46	230	300	-0.8660		+0.5000				
47	235	330	-0.5000		0.8660				
48	240	360	0.0000		1.0000				
49	245	30	+0.5000		0.8660				
50	250	60	0.8660		0.5000				
51	255	90	1.0000		0.0000				
52	260	120	0.8660		-0.5000				
53	265	150	0.5000		-0.8660				
54	270	180	0.0000		-1.0000				
55	275	210	-0.5000		-0.8660				
56	280	240	-0.8660		-0.5000				
57	285	270	-1.0000		0.0000				
58	290	300	-0.8660		+0.5000				
59	295	330	-0.5000		0.8660				
60	300	360	0.0000		1.0000				
61	305	30	+0.5000		0.8660				
62	310	60	0.8660		0.5000				
63	315	90	1.0000		0.0000				
64	320	120	0.8660		-0.5000				
65	325	150	0.5000		-0.8660				
66	330	180	0.0000		-1.0000				
67	335	210	-0.5000		-0.8660				
68	340	240	-0.8660		-0.5000				
69	345	270	-1.0000		0.0000				
70	350	300	-0.8660		+0.5000				
71	355	330	-0.5000		0.8660				
72	360	360	0.0000		1.0000				

TABLE XVa
SEVENTH HARMONIC ($n=7$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
0	0	0	0.0000		1.0000				
1	5	35	0.5736		0.8192				
2	10	70	0.9397		0.3420				
3	15	105	0.9659		-0.2588				
4	20	140	0.6428		-0.7660				
5	25	175	0.0872		-0.9962				
6	30	210	-0.5000		-0.8660				
7	35	245	-0.9063		-0.4226				
8	40	280	-0.9848		+0.1737				
9	45	315	-0.7071		0.7071				
10	50	350	-0.1737		0.9848				
11	55	25	+0.4226		0.9063				
12	60	60	0.8660		0.5000				
13	65	95	0.9962		-0.0872				
14	70	130	0.7660		-0.6428				
15	75	165	0.2588		-0.9659				
16	80	200	-0.3420		-0.9397				
17	85	235	-0.8192		-0.5736				
18	90	270	-1.0000		0.0000				
19	95	305	-0.8192		0.5736				
20	100	340	-0.3420		0.9397				
21	105	15	+0.2588		0.9659				
22	110	50	0.7660		0.6428				
23	115	85	0.9962		0.0872				
24	120	120	0.8660		-0.5000				
25	125	155	0.4226		-0.9063				
26	130	190	-0.1737		-0.9848				
27	135	225	-0.7071		-0.7071				
28	140	260	-0.9848		-0.1737				
29	145	295	-0.9063		+0.4226				
30	150	330	-0.5000		0.8660				
31	155	5	+0.0872		0.9962				
32	160	40	0.6428		0.7660				
33	165	75	0.9659		0.2588				
34	170	110	0.9397		-0.3420				
35	175	145	0.5736		-0.8192				
36	180	180	0.0000		-1.0000				

TABLE XVb
SEVENTH HARMONIC ($n=7$)

u	$u \Delta x = x^\circ$	nx°	$\sin (nx)$	$f(x)$	$\cos (nx)$	$f(x) \sin (nx)$		$f(x) \cos (nx)$	
						pos.	neg.	pos.	neg.
37	185	215	-0.5736		-0.8192				
38	190	250	-0.9397		-0.3420				
39	195	285	-0.9659		+0.2588				
40	200	320	-0.6428		0.7660				
41	205	355	-0.0872		0.9962				
42	210	30	+0.5000		0.8660				
43	215	65	0.9063		0.4226				
44	220	100	0.9848		-0.1737				
45	225	135	0.7071		-0.7071				
46	230	170	0.1737		-0.9848				
47	235	205	-0.4226		-0.9063				
48	240	240	-0.8660		-0.5000				
49	245	275	-0.9962		+0.0872				
50	250	310	-0.7660		0.6428				
51	255	345	-0.2588		0.9659				
52	260	20	+0.3420		0.9397				
53	265	55	0.8192		0.5736				
54	270	90	1.0000		0.0000				
55	275	125	0.8192		-0.5736				
56	280	160	0.3420		-0.9397				
57	285	195	-0.2588		-0.9659				
58	290	230	-0.7660		-0.6428				
59	295	265	-0.9962		-0.0872				
60	300	300	-0.8660		+0.5000				
61	305	335	-0.4226		0.9063				
62	310	10	+0.1737		0.9848				
63	315	45	0.7071		0.7071				
64	320	80	0.9848		0.1737				
65	325	115	0.9063		-0.4226				
66	330	150	0.5000		-0.8660				
67	335	185	-0.0872		-0.9962				
68	340	220	-0.6428		-0.7660				
69	345	255	-0.9659		-0.2588				
70	350	290	-0.9397		+0.3420				
71	355	325	-0.5736		0.8192				
72	360	360	0.0000		1.0000				

INDEX

A

A (symbol), 40
 Abampere, 38
 Abohm, 38
 Abscissa, 21
 A.C.S.R., tables of:
 reactance, 361
 resistance, 358
 susceptance, 363
 Addition
 complex numbers, 5
 sign of, 2
 sine functions, 27
 vectors, 16
 Admittance, 41, 92-94
 equivalent, 216
 operator, 93
 surge, 323
 triangle, 93, 94, 96
 Air, 45, 52, 55
 Alternating, Emf., 58, 287, 291
 Alternating, simple-electric circuits, 58-71
 Alternator, 59, 225
 Ambient medium, 49, 137
 Ampere, 39, 40
 Ampere turn, 38, 41
 Amplitude, 22, 60, 196, 204
 Amplitude factor, 26, 196
 Analysis, harmonic, 168-203
 tables for, 364-377
 Analyzer, 189
 Analyzing a wave, 208
 Angle, electrical, 60
 initial, 26, 204
 phase, 26
 plane, 40
 rotation by an, 10
 Angles, complimentary, 109
 Angular velocity, 30, 31
 Anomalous, dielectric, 327
 Anomaly, 82
 Anti-parallel vectors, 15
 Apparent power, 159, 218

Area, 40
 Argument, 4, 21, 23
 increased, 10
 of vector, 32
 Armature, 51, 59, 225
 variation in current of, 135
 Arrow, 14
 Asymmetrical function, 175
 Attenuation constant, 331
 Average power, 147, 149
 Average value,
 periodic function, 24, 191-192
 in space, 320
 Axis, of imaginaries, 2
 of reals, 2

B

B (symbol), 41
 B-H curve, 51
 Balanced polyphase circuits, 225-259
 Balanced sources, 227
 Bisymmetry, 23, 24
 Branch, 128
 Bridge circuit, 133, 134, 145

C

C (symbol), 41
 Capacitance, 41, 46, 52-55
 circuit of, 66-68
 distributed, 317, 318
 Capacitances
 in parallel, 102
 in series, 86
 Capacitive reactance, 66, 67
 Capacitive susceptance, 66, 67
 Capacitor, 52, 54, 55
 Centimeter, 40
 cube, 42, 48
 Characteristics, concentrated (lumped),
 46-57
 distributed, 317-319
 in parallel, 91-105
 in series, 72-90
 in series-parallel, 121-134

- Characteristics, self-contained, 135
 variable, 106-120
 Charge, 66
 curve, 302
 electric, 40, 52-54
 line, 320
 oscillatory, 304
 Charged, capacitor, 54
 Charging current, 52, 329
 Charging Emf., 54
 Circle, 109
 Circuit, equivalent, 123-124
 Circuit of:
 capacitance, 66-68, 211-213
 distributed characteristics, 317-343
 inductance, 64-66, 208-210
 resistance, 62-64, 206-208
 Circuit, parallel:
 G and C , 94-96
 G and $\mathcal{L}$, 91-93
 G , $\mathcal{L}$, and C , 99-101, 216-217
 $\mathcal{L}$ and C , 96-98
 Circuit, series:
 $\mathcal{L}$ and C , 79-81
 R and C , 76-79
 R and $\mathcal{L}$, 72-75, 213-214
 R , $\mathcal{L}$, and C , 83-84, 214-215
 Circuits, coupled, 135-146
 polyphase, balanced, 225-257
 equivalent, 239-241
 Circular form, 3
 Circular functions, tables of, 350-351
 Circulating current, 232, 246
 Cis operator, 4
 Clockwise direction, 32, 49, 332
 Coefficient, coupling, 140
 mutual induction, 41
 self-induction, 41, 50
 Coils, insulated, 59
 Combination, Emf. impedance, 128
 non-convertible, 124
 Commutation, 135
 Complex admittance, 93
 Complex impedance, 75
 Complex numbers, 1-13
 addition, 5
 argument, 4
 circular form, 3
 definition, 1
 division, 1
 Complex numbers, equality, 5
 exponential form, 4
 forms of expressing, 3-5
 graphic representation, 2
 imaginary part, 2
 imaginary powers, 8
 j -part, 2
 modulus, 4
 multiplication, 6
 notation, 1
 orthogonal form, 3
 periodicity, 8
 polar form, 4
 powers, 7, 8
 quotient of, 7
 real part, 2
 representation of vectors by, 15
 roots of, 7
 subtraction, 5
 trigonometric form, 3
 vectorial form, 3
 Complex product, 17
 Complex quantities, 118
 See also Complex numbers, 1-13
 Complex quotient, 19
 Condenser, 52, 55
 Conditional equivalence, 269
 Conductance, 41, 46
 Conductivity, 41, 48
 Conductor, 58-60
 Conjugate vectors, 15
 Conservation of energy, 327
 Constant voltage, operation at, 327
 Contact, perfect, 54
 Continuous current, 315
 Continuous Emf., 77, 83
 Conversion, 124
 Coördinates, rectangular, 3
 Coplanar vectors, 14, 32, 344
 Copper conductors, table of reactance of, 360
 table of resistance of, 358
 table of susceptance of, 362
 Corona, 52
 Cosine, functions, 33
 hyperbolic, 323
 Coulomb, 40
 Counter-clockwise direction, 32, 49, 331
 Coupled circuits, 135-146
 equivalence in, 141

Coupled circuits, loosely coupled, 140
 resonance in, 142
 Coupling, capacitance, 135, 136
 coefficient, 140
 combination, 135, 136
 inductive, 135, 136
 mutual inductance, 135, 136
 resistance, 135, 136
 types of, 135
 Cross product, 18
 Cubic centimeter, 40, 48
 Current, branch, 129
 circulating, 232, 233
 constant, 106
 curve, 302
 density, 40
 equations, 130
 external, 62
 flow of, 47, 48, 56
 interrupted, 53
 momentary, 53
 non-sine, 204
 notation, 40, 61-62
 oscillatory, 305
 resonance, 98, 126
 triangle, 93, 96
 unit of, 40
 variable, 106-115
 Curve, B-H, 51
 Fourier's series for, 206
 Cycle, 41
 notation for, 24
 of flux, 61
 Cycles per second, 61
 Cyclic transformation of subscripts, 230,
 264, 270
 Cylinder, hollow, 56

D

D (symbol), 40
 Damped, 305
 Damping factor, 305
 Daraf, 41
 D.C. component of Emf., 214
 D.C. machine, 175
 Dead-beat case, 303, 308
 Decrement, 305
 Degree, 40
 Delta, 229, 233, 239
 Density of charge, 40

Dependent variable, 21
 Derived units, 39
 Determinants, 265, 346
 Dielectric, 52
 homogeneous, 54
 perfect, 52, 54
 permittivity, 39, 40
 Differential equation, 286, 293, 346, 347
 Dimension, physical, 37-45
 Direction angle, 15, 16, 32
 Direction of vector, 14
 Discharged capacitor, 54
 Discontinuities, 170
 Distance, variation of E and I with, 331-
 332
 Distributed characteristics in circuits,
 317-343
 capacitance, 317, 318
 inductance, 317, 318
 leakance, 317, 318
 resistance, 317, 318
 Division, of complex numbers, 7
 of sine functions, 29
 of vectors, 19
 Dot product, 18
 Double-energy, 282, 301-316
 Double-frequency, 29

E

E (symbol), 40
 E variable, 111-114
 Earth, 52
 Effective value, 24-26, 193-196
 Efficiency, 340
 Elastance, 41, 54
 Elastivity, 41
 Electric, charge, 54
 Electric circuit, characteristics, 46-57
 definition, 61
 Electric current as fundamental unit, 39,
 45
 Electric intensity, 41
 Electric machine, 116
 Electric resistance as fundamental unit,
 39, 45
 Electric stress, 41
 Electric transients, 281-316
 Electrical angle, 60
 Electrical radians, 60
 Electricity, quantity of, 40, 53

- Electricity, transfer of, 53
 - Electromagnetic energy, 51
 - Electromagnetic induction, 42, 58
 - Electromagnetic system, 38, 39, 45
 - Electromagnetic wave, 45
 - Electrostatic flux, 54
 - Electrostatic system, 39, 45
 - Ellipse, as locus, 116-117
 - Emf., 40, 46
 - alternating, 58
 - at source terminals, 61
 - charging, 53
 - combinations, 121, 126
 - continuous, 50, 83
 - equations, 130
 - induced, 49
 - local, 72
 - non-sine, 204
 - notation, 61-62, 227
 - of mutual induction, 137
 - of self-induction, 52
 - sine, 58, 77
 - unit of, 42
 - unvarying, 42, 46
 - Emf.-Impedance, combination, 128, 218
 - Energy, conservation of, 327
 - radiated, 306
 - returned (max. rate), 164
 - stored, 164, 281
 - Energy and power, 147-167
 - Energy in circuits, coupled, 163
 - of capacitance, 44, 55, 154
 - of inductance, 48, 51, 153
 - of non-sine waves, 220
 - of resistance, 152
 - series, 156, 158
 - Engineer, 37, 38
 - Entities, 1, 2
 - Equal vectors, 14
 - Equality, 5, 170
 - Equations, dependent, 129
 - dimensional, 42
 - independent, 129, 235
 - simultaneous, 9
 - Equivalence, 121, 122, 141
 - Equivalent circuit, 68, 83, 123, 124
 - mesh and star, 239-241, 267-272
 - Equivalent for value of:
 - admittance, 126
 - capacitance, 90
 - Equivalent for value of: impedance, 90, 215
 - inductance, 90
 - reactance, 90, 210, 213
 - resistance, 90
 - susceptance, 97, 98
 - Equivalent sine function, 198, 210, 217
 - Equivalent star load, 240, 241, 268-271
 - Equivalent star source, 240, 267
 - Erg, 38
 - Exponent, 43
 - Exponential form, 3, 4
 - Exponential functions, 287
 - tables of, 352-357
 - External circuit, 61, 72, 127
- F
- F* (symbol), 41
 - Farad, 41, 55
 - Faraday, 58
 - Faraday's dynamo, 68
 - Faraday's law, 42, 45, 59
 - Field, revolving, 59
 - Field current, 135
 - Fifth harmonic, table for, 372-373
 - First harmonic, 204
 - Flux, 50
 - density, 14, 38, 58, 60
 - in air-gap, 202
 - linkage, 50, 139
 - linking a circuit, 49, 50
 - linking current, 50, 136, 137
 - magnetic, 41
 - mutual, 137
 - stationary, 59
 - time-rate of change of, 49
 - wave, abscissa scale of, 60
 - Force, 14
 - Forced oscillations, 335
 - Form factor, 26, 196
 - Forms of complex numbers, 3
 - Four-phase, machine, 227
 - Fourier, 204
 - Fourier's series, 168-203, 205, 345
 - constant term in, 171
 - curves analyzed into, 206
 - forms of, 189
 - functions expressed by, 183
 - graphic meaning of terms in, 171

Fourier's theorem, 171-172
 Fourth harmonic, table for, 370-371
 Free oscillations, 301, 335
 Frequency, 41, 61
 resonant, 82, 125
 revolving vector, 30
 Fringing, 54
 Fundamental, harmonic, 168, 204, 206
 Table for, 364-365
 Fundamental units, 39, 40
 Function, 170, 182-189
 aperiodic, 185
 Functions, circular, tables of, 350-351
 explicitly defined, 182
 exponential, tables of, 352-357
 graphically defined, 182, 186-189
 hyperbolic, tables of, 352-357
 non-periodic, 185
 non-sine, 204
 trigonometric, tables of, 350-351

G

G (symbol), 41, 46
 Galvanometer, 53, 54
 Gaseous conductor, 47
 Gauss, 38
 General solution, 346-348
 Generation, of sine Emfs., 59
 of sine functions, 30
 Generator voltage, 328, 330
 Gilbert, 38
 Gram, 40
 Graphic addition, of sine functions, 27
 of vectors, 16
 Graphic representation, of complex
 numbers, 2
 of sine functions, 21
 of vectors, 14-15
 Graphic subtraction, of sine functions, 28
 of vectors, 17
 Graphs, sine functions, 21, 22

H

Harmonic, 204
 abscissa scale of, 168
 acting alone, 208, 214
 analysis, 168-203
 tables for, 364-377
 analyzer, 189
 Harmonics, cosine, 171

Harmonics, even, 175
 in three-phase circuits, 241-244
 non-triplen, 242, 243
 odd, 177, 179, 241
 order of, 168
 scale of, 168
 sine, 171
 triplen, 242, 243
 Heat, 47
 Helix, 50
 Henry, 41, 42
 Hyperbolic cosine, 323
 Hyperbolic functions, tables of, 352-357
 Hyperbolic sine, 323
 Hystero-viscosity, 54

I

I (symbol), 40
 I variable, 107, 109, 110
 Identity, 5, 170
 Imaginary, 1, 9
 Impedance, 41, 73, 77
 coil, 87
 complex, 75
 equivalent, 215
 referred to primary, 142
 referred to secondary, 142
 operator, 75
 surge, 323
 triangle, 75, 76, 79
 Impedance-Emf. combinations, 128
 Independent equations, 130, 235
 Induced Emf., 51
 Induced voltage, 58, 65, 139
 Inductance, 41, 42, 48-52
 circuit of, 64-66
 distributed, 317, 318
 Inductances in parallel, 102
 in series, 86
 Induction, electromagnetic, 58
 motor, 106
 Inductive, reactance, 65, 66
 Inductive susceptance, 64, 66
 Inertia, 48, 303
 Inflection point, 304
 Initial angle, 26-28, 31, 204
 Initial value, 22, 23
 Instantaneous power, 147-149, 218
 maximum, 164
 minimum, 164

Instantaneous value, 22, 206
 Integer, real, 1
 Integration constant, 77
 Internal circuit, 61
 International commission, 219
 Interval, 23

J

J (symbol), 1
 as operator, 9, 10
J-axis, 2
J-number, 1
 Joule, 38, 41
 Joule's law, 45, 46, 47
 Junction, 128, 129, 130

K

Kirchhoff, current law of, 91-93
 Emf. law of, 72, 73, 79, 126

L

L (symbol), 40
 Lag, 26
 Lead, 26
 Leakance, 317, 319
 Leaky current, 329
 Length, 39, 40
 of vector, 15, 32
 Lenz, 50
 Life, 82
 Limitations, of concept of inductance, 50
 of Fourier's Theorem, 171
 of Ohm's law, 46-47
 Limiting case, 303, 306, 308
 Limits, 24, 25
 Line currents, 235, 238
 Line drop, 328
 Line impedance, 263, 265
 Line loss, 340
 Line voltage, 229-232, 237, 259
 Linear differential Eq., 301, 346, 347
 Lines of force, 49
 Liquid conductor, 47
 Load, 234
 currents, 238, 329
 end, 322
 impedance, 235, 263
 K.V.A., 330
 power factor, 330
 Loading, polyphase sources, 234

Loci, 323-327
 Locus, 106, 333
 given, 115
 indefinite, 110, 114
 of current, 108, 110
 Logarithm, 4, 8
 Logarithmic spiral, 332
 Loop, 128-129, 232-233
 single, 130
 Lumped characteristics, 317

M

M (symbol), 40
 Machine, electrical, 135
 Maclaurin's theorem, 11
 Magnetic field, 48, 49
 Magnetic intensity, 41
 Magnetic permeability, 39, 40, 49, 137
 Magnetomotive force, 38, 41
 Magnitude, 38
 of vector, 14, 15
 Mass, 14, 39, 40
 Material, 52
 Mathematician, 172
 Maximum value, 22
 Maxwell, 38, 44
 Mechanical radians, 60
 Mesh connection, 229
 Mesh load, 234, 265
 Mesh source, 232-234, 265
 Mesh-mesh circuit, 237-238, 265-266
 Mesh-star circuit, 236-237, 264-265
 Metallic conductor, 47, 52
 Mho, 41, 64
 Mile, 322
 Minimum K.V.A., 330
 Mmf., 41, 200-201
 Modulus, 4
 Multiplication, of complex numbers, 6
 of sine functions, 28
 of vectors, 17
 Mutual flux, 137
 Mutual inductance, 137-138, 140
 Mutual reactance, 141

N

Nature, 37
 Negative, angle, 32
 Negative phase sequence, 244

Negative vectors, 14
 Neutral, in polyphase circuits, 245
 oscillating, 246
 Non-anomalous, dielectric, 54
 Non-convertible, combination, 124
 Non-inductive, circuit, 51
 Non-interconnected, circuit, 234, 262
 Non-sine current, 204
 Non-sine Emf., 204
 Non-sine waves, 169, 198, 205, 272
 Non-triplen, 242, 244, 249
 Notation, double-subscript, 62
 legend of, 38, 40-41
 Notation for:
 complex numbers 1, 2
 complex product, 18
 cross product, 18
 current, 61-62
 dot product, 18
 Emf., 61-62
 operator, 16
 vector, 14
 vector product, 18
 Number, 41
 Numbers, complex, 1-13
 imaginary, 1
 real, 1, 2, 3
 Numerical relations, 37

O

Odd harmonics, 241
 Ohm, 38, 39, 41
 Ohm's law, 45-47, 54
 Operator, 9, 16, 17
 Order, 17, 168
 Ordinate, 21
 Origin, 14
 Orthogonal form, 3
 Oscillating neutral, 246
 Oscillations, 309
 forced, 334, 339
 frequency of, 315
 natural, 314
 Oscillatory case, 303, 304, 308
 Oscillatory charge, 304
 Oscillatory current, 305
 Oscillograph, 121
 connections, 61
 Out of phase, meaning of, 27

P

P (symbol), 41
 Parallel circuit, 91-105
 definition, 91
 general expressions for, 101
 Parallel circuit consisting of:
 G , $\mathcal{L}$, and C , 99-101
 G and C , 94-96
 G and $\mathcal{L}$, 91-93
 $\mathcal{L}$ and C , 96-98
 several characteristics, 101-102
 variable characteristics, 106-120
 Parallel vectors, 14
 Partial resonance, 146
 Path, 61
 Period, 30
 Periodic function, 23, 190-192
 Periodic interchange of energy, 301
 Periodic, non-sine wave, 169, 170, 204
 Periodicity of:
 j^n , 1
 sine functions, 23
 Permeability, 50, 51
 Permittance, 41
 Permittivity, 55
 Phase, 32, 196-198, 225
 angle, 26, 27, 31, 122
 currents, 235
 definition, 26
 displacement, 225
 out of phase, meaning of, 27
 sequence, 244, 250
 voltages, 229-232, 259
 Phenomena of electric circuits, 37, 45,
 61
 Physical dimensions, 37-45
 Physical laws, 38, 42, 43
 Physical quantities, 37
 Physical units, 37
 Physicist, 37, 38
 Pi value, 22-23
 Plate, 53
 Polar coördinates, 21, 117
 Polar form, of expressing complex num-
 bers, 3, 4
 Pole, 59, 60, 61
 Polygon, 20
 Polyphase, alternator, 225
 Polyphase Emfs., 227
 Polyphase loads, 268, 269

Polyphase sources, 227-229, 234, 265

Polyphase transformations, 258

Polyphase circuits:

balanced, 225-257

equivalent, 239-241

harmonics in, 241-247

mesh-mesh, 237-238

mesh-star, 236-237

non-interconnected, 234-235

power in, 247-250

star-mesh, 239

star-star, 235-236

types of, 234

unbalanced, 258-280

equivalent, 267-272

harmonics in, 272

mesh-mesh, 265-266

mesh-star, 264-265

non-interconnected, 262

power in, 272-275

star-mesh, 266-267

star-star, 263-264

types of, 262

Positive angle, 32

Positive phase-sequence, 244

Potential drop in:

capacitance, 66

inductance, 65

resistance, 63

Potential gradient, 14, 41

Power, 147-167

apparent, 159, 273

average, 147

complex, 161

electric, 14, 41

general expression for, 148, 149

instantaneous, 147

reactive, 149, 160

transmission, 317

wattless, 149, 160

Power factor, 41, 149, 159-160

non-sine waves, 218-219

polyphase circuits, 273

Power in:

circuit with C , 153-154

circuits with distributed characteristics, 340

circuit with $\mathcal{L}$, 152-153

circuits with non-sine waves, 218-219, 275

Power in: circuit with R , 150-152

coupled circuits, 161-163

multi-Emf. circuits, 158-159

polyphase circuits, 247-250, 272-275

series circuit with $\mathcal{L}$ and C , 154-156

series circuit with R and $\mathcal{L}$, 156-158

Powers and roots, 7, 8

Practical system, 38, 39, 45

units in, 40, 41

Primary, 135, 137, 142

Principle of superposition, 131

Product, 6, 17, 18, 29

Projections, 15, 16, 30

Properties, 23

Propagation constant, 323

Proximity, 135

Pull, magnetic, 45

Q

Q (symbol), 40

Quadrant, 3

Quantity, 53

Quarter-phase, 259

Questionnaire, 160

Quotient, 7, 19, 30

R

R (symbol), 40, 46

R , variable, 107, 199, 113, 114

R -axis, 2

Radians, 8, 40, 60

Radiated energy, 306

Radio circuits, 135, 204

Radius vector, 21

Reactance, 41, 68, 210

capacitive, 66, 67

equivalent, 80, 125

inductive, 64

mutual, 141

of A.C.S.R., table of, 361

of copper conductors, table of, 360

Reactive power, 149, 160, 219

Real quantities, 1, 2

Receiver, current, 323

Emf., 323

end, 322, 323, 340

Reciprocal vectors, 15

Rectangular coördinates, 2, 3, 21

Rectified sine wave, 175, 182, 202

Regulator, voltage, 297

Relation between $\mathfrak{M}$ and $\mathcal{L}$, 140
 Reluctivity, 41
 Representation by vector, 196–198
 Resilient rod, 335
 Resistance, 40, 46–48
 circuit of, 62–64
 distributed, 317, 318
 equivalent, 125
 measuring, 56
 nature of, 46
 of A.C.S.R., table of, 358
 of copper conductors, table of, 358
 Resistivity, 41, 48
 Resonance, 68, 125
 complete, 99
 concept of, 82
 current, 98, 99, 217
 partial, 99, 142, 146
 perfect, 82
 voltage, 81, 82, 215
 Resonant frequency, 82, 125, 126
 in distributed characteristics, 336–340
 Resonant points, 143
 Revolutions, 61
 Right-hand rule, 19
 Roots, of complex numbers, 11
 Rotary converter, 226
 Rotating operator, 9
 Rotor reactance, 106
 Rumanian questionnaire, 160, 219

S

S (symbol), 41
 Scalar quantities, 14, 18
 Scale of harmonics, 206, 208
 Science, progress of, 37
 Scott connection, 258
 Screw, 49
 Second, 40
 Second harmonic, table for, 366–367
 Secondary, 135, 137, 142
 Self-inductance, 50, 138
 Self-induction, Emf., of, 51, 52
 Semicircle, 109
 Sender end, 323
 Series circuit, 72–90, 106–120
 with $\mathcal{L}$ and C , 79–81
 with R and C , 76–79
 with R and $\mathcal{L}$, 72
 with R , $\mathcal{L}$, and C , 83–84

Series circuit, with several R , $\mathcal{L}$, and C ,
 85–86
 Series, Fourier's, 168–203
 Series-parallel circuits, 121, 122, 217
 characteristics, 121, 122
 combinations, 121–134
 Seventh harmonic, table for, 376–377
 Short-circuit transient, 307–309, 313–314
 Sign in mathematical operations, addi-
 tion, 2
 subtraction, 2
 summation, 170
 Simultaneous equations, 9
 Simultaneous variation, 110, 114
 Sine curve, 28
 Sine Emfs., 58, 59
 Sine functions, 21–36
 addition, 27
 amplitude, 22
 argument, 23
 average value, 24
 bissymmetry, 23, 24
 damped, 305
 decremental, 305
 definition, 21
 division, 29
 effective value, 25
 generation, 30
 graphic representation, 21–22
 initial angle, 23
 initial value, 22
 instantaneous value, 22
 multiplication, 23
 periodicity, 23
 π values, 23
 properties, 23
 representation by vector, 31–33
 subtraction, 28
 zero value, 23
 Sine, hyperbolic, 323
 Sine voltage, 61
 Sine wave, 21, 175, 184
 Sines of multiples of x , 168
 Single-energy transient, 281–300
 Sixth harmonic, table for, 374–375
 Skin-effect, 207, 209, 346
 Slope, 23
 Slot, 59, 135, 225
 Smooth, wave, 208, 210
 Space waves, 332, 336–340

Spark, 53
 Sphere, 52
 Spiral, elliptical, 332
 logarithmic, 332
 Square centimeter, 40
 Star connection, 229
 Star load, 234, 265
 Star source, 229-232
 Star voltages, 236
 Star-mesh circuit, 239, 266
 Star-star circuit, 235-236, 263-264
 Statampere, 38
 Stationary vectors, 31, 32
 Statohm, 38
 Steady current, 287
 Steady state, 31, 107, 291
 Steady vs. transient, 115
 Steps in conversion, 123
 Storehouse, 51
 Subscripts, cyclic, 62, 230, 264
 Subtraction, of complex numbers, 5, 6
 of sine functions, 28
 of vectors, 17
 Summation, replacing integral by, 186
 sign of, 170, 205
 Superposable currents, 131
 Superposition, principle of, 131
 Surge, 323
 Susceptance, 41, 66, 67, 96
 equivalent, 125
 of A.C.S.R., tables of, 363
 of copper conductors, tables of, 362
 Switch, 53
 Switching, 286, 292
 Symbol, 3
 Symmetrical components, 262
 Symmetrical functions, 172
 Symmetrical voltages, 263
 Symmetry, 172-175
 effect of, 175-182
 table of, 176-177
 Synchronous condenser, 327-330
 Synchronous motor, 115, 127
 System, 2, 46

T

T (symbol), 40
 T connection, 258-260
 Table of:
 exponential functions, 352-357

Table of: harmonic analysis, 364-377
 hyperbolic functions, 352-357
 physical units, 40, 41
 reactance, 360-361
 resistance, 358-359
 susceptance, 262-263
 symmetry, 176-177
 Taps, 258
 Telephone, 204
 Temperature, 47
 Terminals, 229
 Terminus, locus of, 106
 of vector, 14
 Theorems, I, 344
 II, 344
 III, 345
 IV, 346
 V, 346
 VI, 347
 Theoretical physics, 44
 Thesis, 269
 Third harmonic, table for, 368-369
 Three-phase, 227
 Three-phase alternator, 226
 Three-phase balanced circuits, 234-250
 Time, 39, 40
 constant, 283, 287, 291, 294
 Transformations, polyphase, 258-262
 Transformer, 126, 135, 145, 146
 Transient, 106, 284
 inevitability of, 312-313
 Transients, double energy, 282, 301-316
 single energy, 281-300
 Transients in series circuits of:
 R and C , 290-296
 R and $\mathcal{L}$, 282-290
 R , $\mathcal{L}$, and C , 302-314
 Transmission, electric power, 236, 317,
 340
 Transmission line, 17, 335
 Triangles, admittance, 93, 96
 current, 93, 96
 impedance, 75, 79
 power, 160
 voltage, 75, 79
 Trigonometric form, 3, 4
 Trigonometric functions, tables of, 350-
 351
 Triplen harmonics, 242, 244-246
 power due to, 249

Tubes of force, 48
 Turbine, 299
 Turbo-alternator, 299
 Turns, 50
 Two-phase alternator, 225, 226, 227
 Two-phase transformation, 258-262

U

U (symbol), 40
 Unbalanced polyphase circuits, 258-280
 Unique equivalence, 268
 Universal equivalence, 269, 271-272

V

V (symbol), 40
 V connection, 260
 Vacuum, permittivity of, 55
 Variation of characteristics, 106, 115
 Vectorial angle, 21
 Vectors, 14-20
 addition, 16
 conjugate, 15
 coplanar, 14
 definition, 14
 direction angle, 15
 division, 19
 equality, 14
 graphic representation, 14
 length, 15
 multiplication, 17-19
 negative, 15
 notation for, 14
 origin of, 14
 product of, 18
 reciprocal, 15
 representation by complex numbers, 15
 representation of sine functions by, 32
 revolving, 30
 stationary, 31
 subtraction, 17

Vectors, terminus of, 14
 unit, 15
 Velocity, angular, 40
 linear, 40
 of light, 44
 relative, 58, 60
 Volt, 38, 40
 Volt-ampere, 219
 Voltage, induced, 52, 225
 resonance, 81, 82, 126
 sine, 61
 triangle, 75, 79
 Volume, 40, 42

W

W (symbol), 41
 Watt, 41
 Wattless power, 149, 160
 Wattmeter, 147
 Wave, non-sine, 169, 205
 rectified, 201
 Wavelength, 336-340
 Waves in space, 332
 Weber, 38, 41
 Windings, independent, 225
 Wire, 52

X

X (symbol), 41
 X , variable, 107, 110, 111, 114

Y

Y (symbol), 41, 92
 Y connection, 229, 230
 Yrneh, 41

Z

Z (symbol), 41
 Zero value, 22, 23



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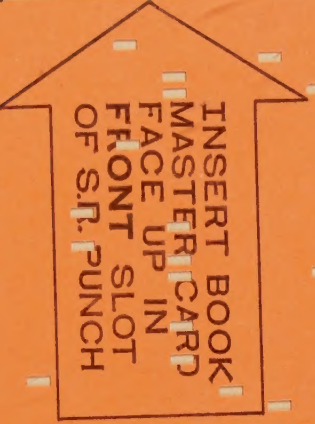
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